Practice Problems 1
Introduction to Modal Logic

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1. Prove that $\dfrac{\phi \to \psi}{\Box\phi \to \Box\psi}$ is a valid rule of inference (in the class of all frames).

   - Prove that if $\vdash_{\mathbf{K}} \phi \to \psi$, then $\vdash_{\mathbf{K}} \Box\phi \to \Box\psi$

2. Prove that $\dfrac{\phi \to \psi}{\Diamond\phi \to \Diamond\psi}$ is a valid rule of inference (in the class of all frames).

   - Prove if $\vdash_{\mathbf{K}} \phi \to \psi$ then $\vdash_{\mathbf{K}} \Diamond\phi \to \Diamond\psi$.
   - Try proving if $\vdash_{\mathbf{K}} \phi \to \psi$ then $\vdash_{\mathbf{K}} \Diamond\phi \to \Diamond\psi$ *without using the Dual axiom* — **Remember,** $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$ **is short for** $\neg\Diamond\neg(\phi \to \psi) \to (\neg\Diamond\neg\phi \to \neg\Diamond\neg\psi)$.

3. Prove $\vdash_{\mathbf{K}} \Box(\phi \wedge \psi) \leftrightarrow (\Box\phi \wedge \Box\psi)$

4. Prove a general substitution theorem:

   $$\text{If } \vdash_{\mathbf{K}} \phi \leftrightarrow \psi \text{ then } \vdash_{\mathbf{K}} \Theta \leftrightarrow \Theta'$$

   where Θ' is Θ with every occurence of ϕ replaced by ψ.

5. The axiom system **K4** is obtained by adding the axiom $\Diamond\Diamond p \to \Diamond p$ to **K** and **S4** is **K4** with the additional axiom $p \to \Diamond p$.

   - Show that $\nvdash_{\mathbf{S4}} p \to \Box\Diamond p$ (**Hint: find a class of frames for which S4 is sound**).
   - The axiom system **S5** is **S4** plus the axiom $p \to \Box\Diamond p$. Show that $\vdash_{\mathbf{S5}} \Diamond p \to \Box\Diamond p$. The following is a sketch of the derivation.
     (a) $p \to \Box\Diamond p$ (Axiom)
     (b) $\Diamond p \to \Box\Diamond\Diamond p$ (Substitution – replace p with $\Diamond p$)
     (c) $\Diamond\Diamond p \to \Diamond p$ (Axiom)
     (d) $\Box\Diamond\Diamond p \to \Box\Diamond p$ (Rule of inference from 1.)
     (e) $\Diamond p \to \Box\Diamond p$ (Propositional Reasoning using b,d)

- If $\Box\phi$ is interpreted as "some agent knows ϕ", what is the intended interpretation of $\Diamond p \rightarrow \Box\Diamond p$ (it may be easier to write this formula as $\neg\Box p \rightarrow \Box\neg\Box p$).