# Practice Problems 5
# Introduction to Modal Logic

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**Properties of the reflexive transitive closure:** Let $R$ and $S$ be a binary relations on $W$ and $\iota = \{(s,s) \mid s \in W\}$ the identity relation. We define $R \circ S = \{(u,w) \mid \exists v\ (u,v) \in R \text{ and } (v,w) \in S\}$. Define $R^n$ inductively as follows: $R^0 = \iota$, $R^{n+1} = R \circ R^n$. Finally define $R^* = \cup_{n \geq 0} R^n$ and $R^+ = \cup_{n \geq 1} R^n$. The following exercises will help you become familiar with these operations:

1. Prove the following set equivalences:

   (a) Prove that if $m, n \geq 0$, $R^n \circ R^m = R^{n+m}$

   (b) Prove that $R^* \circ R = R \circ R^*$

   (c) $R^{**} = R^*$

   (d) $R \subseteq R^*$

   (e) $R^* \circ R^* = R^*$

   (f) $\iota \cup (R \circ R^*) = R^*$

2. Prove that $R^+$ is the smallest (in the sense of set inclusion) transitive relation containing $R$ and $R^*$ is the smallest reflexive transitive relation containing $R$.

3. A set operator on a set $W$ is a function $\tau : 2^W \to 2^W$. The operator $\tau$ is **monotone** if $A \subseteq B$ implies $\tau(A) \subseteq \tau(B)$. A set $A$ is a **prefixed point** of $\tau$ if $\tau(A) \subseteq A$ and the lest prefixed point if for all prefixed points $B$ of $\tau$, $A \subseteq B$. Let $\tau$ be the following set operator on $W \times W$, $\tau(X) = \iota \cup (R \circ X)$. Prove that $\tau$ is monotone and that $R^*$ is the least prefixed point.

**Axiomatization of PDL:** Let $P$ be a set of atomic programs and $\Phi$ a set of atomic propositions. Formulas of PDL have the following syntactic form:

$$\phi := p \mid \bot \mid \neg\phi \mid \phi \vee \psi \mid [\alpha]\phi$$

$$\alpha := a \mid \alpha \cup \beta \mid \alpha;\beta \mid \alpha^* \mid \phi?$$

where $p \in \Phi$ and $a \in P$. The semantics was given in class.

Consider the following axiomatization of PDL.

1. Axioms of propositional logic
2. $[\alpha](\phi \to \psi) \to ([\alpha]\phi \to [\alpha]\psi)$
3. $[\alpha \cup \beta]\phi \leftrightarrow [\alpha]\phi \wedge [\beta]\phi$
4. $[\alpha; \beta]\phi \leftrightarrow [\alpha][\beta]\phi$
5. $[\psi?]\phi \leftrightarrow (\psi \to \phi)$
6. $\phi \wedge [\alpha][\alpha^*]\phi \leftrightarrow [\alpha^*]\phi$
7. $\phi \wedge [\alpha^*](\phi \to [\alpha]\phi) \to [\alpha^*]\phi$
8. Modus Ponens and Necessitation (for each program $\alpha$)

Answer the following questions:

1. Prove that the above axiom system is sound, i.e., show that each of the above axiom schemes are valid.
2. Prove that the following formulas are valid:
   (a) $[\alpha^*]\phi \leftrightarrow [\alpha*; \alpha^*]\phi$
   (b) $[\alpha^*]\phi \to \phi$
   (c) $[\alpha^*]\phi \to [\alpha]\phi$
   (d) $[\alpha^*]\phi \leftrightarrow [\alpha^{**}]\phi$
   (e) $\langle\alpha^*\rangle\phi \leftrightarrow \phi \vee \langle\alpha\rangle\langle\alpha^*\rangle\phi$
   (f) $\langle\alpha^*\rangle\phi \leftrightarrow \phi \vee \langle\alpha^*\rangle(\neg\phi \wedge \langle\alpha\rangle\phi)$
3. Consider the *RTC* rule:

   $$\frac{(\phi \vee \langle\alpha\rangle\psi) \to \psi}{\langle\alpha^*\rangle\phi \to \psi}$$

   The importance of this rule is in its relationship with the above valid formula (2 (e)). Note that this formula states that $\langle\alpha^*\rangle\phi$ is a solution to the equation $\phi \vee \langle\alpha\rangle R \to R$ (that is the formula is valid when $R$ is replaced by $\langle\alpha^*\rangle\phi$). The *RTC* rules says that $\langle\alpha^*\rangle$ is the *least* such solution.

(a) Prove that the $RTC$ rule is sound.

(b) Prove that $RTC$ and the induction axiom are interderivable (in the presence of the other axioms and rules of PDL). That is show that by assuming $RTC$ we can deduce the induction axiom and by assuming the induction axiom we can deduce $RTC$ (for this direction it is easier to show that from the induction axiom we can deduce the loop invariant rule: from $\psi \to [\alpha]\psi$ we can deduce $\psi \to [\alpha^*]\psi$, then from this rule we can deduce $RTC$).

**Filtrations for PDL:** For simplicity, consider the test-free version of PDL (that is, we do not consider the ? operator). We first define the **Fischer-Ladner** closure of a formula. A set of formulas $X$ is **Fischer-Ladner Closed** if it is closed under subformulas and

1. If $[\alpha;\beta]\phi \in X$ then $[\alpha][\beta]\phi \in X$.
2. If $[\alpha \cup \beta]\phi \in X$ then $[\alpha]\phi \in X$ and $[\beta]\phi \in X$
3. If $[\alpha^*]\phi \in X$ then $[\alpha][\alpha^*]\phi \in X$

Given a formula $\phi$, let $FL(\phi)$ be the smallest set containing the subformulas of $\phi$ that is Fischer-Ladner closed. Let $\mathfrak{M} = \langle W, \{R_a \mid a \in P\}, V\rangle$ be a model. Let $\phi$ be a formula of PDL. We say $w \sim_{FL(\phi)} v$ iff for each $\psi \in FL(\phi)$, $\mathfrak{M}, w \Vdash \psi$ iff $\mathfrak{M}, v \Vdash \psi$. Let $|w|$ be the equivalence class of $w$ induced by this equivalence relation. Let $\mathfrak{M}^f = \langle W^f, \{R^f_a \mid a \in P\}, V^f\rangle$ be defined as follows:

- $W^f = W/FL(\phi) = \{|w| \mid w \in W\}$
- $|w|R^f_a|v|$ iff $wR_av$ (for an atomic program $a$)
- $V^f(p) = \{|w| \mid w \in V(p)\}$

**Lemma 1 (Filtration Lemma)** *Let $\mathfrak{M}$ be a model and $\mathfrak{M}^f$ defined as above. Suppose that $w$ and $v$ are states.*

1. *For all $[\alpha]\psi \in FL(\phi)$,*
   (a) *If $wR_\alpha v$ the $|w|R^f_\alpha|v|$*
   (b) *If $|w|R^f_\alpha|v|$ and $\mathfrak{M}, w \Vdash [\alpha]\psi$ then $\mathfrak{M}, v \Vdash \psi$*
2. *For all $\psi \in FL(\phi)$, $\mathfrak{M}, w \Vdash \psi$ iff $\mathfrak{M}^f, |w| \Vdash \psi$.*