

**CS 6280 - Multi agent systems**

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**Elimination of Dominated Strategies**

## Strict Dominance: Recap

Consider a game  $(S_1, \dots, S_n, p_1, \dots, p_n)$ .

- A strategy  $s'_i$  **strictly dominates** a strategy  $s''_i$ , or equivalently, a strategy  $s''_i$  is **strictly dominated** by a strategy  $s'_i$  if

$$p_i(s'_i, s_{-i}) > p_i(s''_i, s_{-i})$$

for all  $s_{-i} \in S_{-i}$ .

- A strategy of player  $i$  is **strictly dominant** if it strictly dominates any other of his strategy.

## Example

Consider

			Two		
		L	M	R	
One	T	3,3	3,1	3,2	
	B	2,2	2,4	2,1	

By eliminating all strictly dominated strategies the game is reduced to

			Two	
		L	M	
One	T	3,3	3,1	

Now, strategy M *is* strictly dominated by the strategy L. Eliminating it we obtain

			Two	
		L		
One	T	3,3		

## Conclusion

Rational players **One** and **Two** will play (T,L).

- Why? Common Knowledge of rational behaviour:

**One** knows that **Two** will not play R.

**Two** knows that **One** will not play B.

**One** knows that **Two** knows that **One** will not play B. So **One** knows that **Two** knows that **One** will play T.

...

- How general is this elimination process?
- In the Battle of the Sexes game no strategy (strictly or weakly) dominates another:

		Woman	
		F	B
Man	F	2,1	0,0
	B	0,0	1,2

- Do we keep all Nash equilibria?

## Iterated Deletion

- Given a game  $G := (S_1, \dots, S_n, p_1, \dots, p_n)$  and non-empty sets of strategies  $S'_1, \dots, S'_n$  such that  $S'_i \subseteq S_i$  for  $i \in [1..n]$  we say that

$$G' := (S'_1, \dots, S'_n, p_1, \dots, p_n)$$

is a **subgame** of  $G$  and identify in the context of  $G'$  each payoff function  $p_i$  with its restriction.

- Consider a game  $G := (S_1, \dots, S_n, p_1, \dots, p_n)$  and its subgame  $G' := (S'_1, \dots, S'_n, p_1, \dots, p_n)$ . Let

$$G \rightarrow_S G'$$

when  $G \neq G'$  and for all  $i \in [1..n]$

each  $s''_i \in S_i \setminus S'_i$  is strictly dominated in  $G$  by some  $s'_i \in S_i$ .

**Note:** we do not require that all strictly dominated strategies are deleted.

# Iterated Deletion and Nash Equilibria

## Strict Elimination Lemma

Suppose that  $G \rightarrow_S G'$ . Then  $s$  is a Nash equilibrium of  $G'$  iff it is a Nash equilibrium of  $G$ .

**Proof.** Let

$$G := (S_1, \dots, S_n, p_1, \dots, p_n),$$

and

$$G' := (S'_1, \dots, S'_n, p_1, \dots, p_n).$$

( $\Rightarrow$ ) Suppose  $s$  is not a Nash equilibrium of  $G$ . Then for some  $i \in [1..n]$  and  $s'_i \in S_i$

$$p_i(s'_i, s_{-i}) > p_i(s).$$

Choose  $s'_i$  for which  $p_i(s'_i, s_{-i})$  attains the maximum.  $s'_i$  is eliminated since  $s$  is a Nash equilibrium of  $G'$ . So for some  $s_i^* \in S_i$

$$p_i(s_i^*, s''_{-i}) > p_i(s'_i, s''_{-i}) \text{ for all } s''_{-i} \in S_{-i}.$$

In particular

$$p_i(s_i^*, s_{-i}) > p_i(s'_i, s_{-i}),$$

which contradicts the choice of  $s'_i$ .

## Iterated Deletion and Nash Equilibria, ctd

( $\Leftarrow$ ) For each player the set of his strategies in  $G'$  is a subset of the set of his strategies in  $G$ .

So to prove that  $s$  is a Nash equilibrium of  $G'$  it suffices to prove that no strategy constituting  $s$  is eliminated.

Suppose otherwise. Then some  $s_i$  is eliminated, so for some  $s'_i \in S_i$

$$p_i(s'_i, s''_{-i}) > p_i(s_i, s''_{-i}) \text{ for all } s''_{-i} \in S_{-i}.$$

In particular

$$p_i(s'_i, s_{-i}) > p_i(s_i, s_{-i}),$$

so  $s$  is not a Nash equilibrium of

$$(S_1, \dots, S_n, p_1, \dots, p_n).$$

## Iterated Deletion, ctd

- $G'$  is an outcome of an iterated elimination of strictly dominated strategies from the game  $G$  (IES) if for no game  $G''$ ,  $G' \rightarrow_S G''$ .
- $G$  is solved by an iterated elimination of strictly dominated strategies if in  $G'$  each player has just one strategy.

### Theorem

Suppose that  $G'$  is an outcome of an IES starting in the game  $G$ .

- (i) Then  $s$  is a Nash equilibrium of  $G'$  iff it is a Nash equilibrium of  $G$ .
- (ii) If  $G$  is solved by an iterated elimination of strictly dominated strategies, then the resulting joint strategy is a unique Nash equilibrium of  $G$ .



## Iterated Deletion, ctd

In other words,

- each Nash equilibrium of the initial game **survives** any iterated elimination of strictly dominated strategies,
- each Nash equilibrium of an outcome of an iterated elimination of strictly dominated strategies **is also** a Nash equilibrium of the initial game,
- if a game is solved by an iterated elimination of strictly dominated strategies, then the **resulting joint strategy** is its Nash equilibrium.

### **Proof.**

(i) By the repeated application of the Strict Elimination Lemma.

(ii) Note that  $(s_1, \dots, s_n)$  is a unique Nash equilibrium of the game  $(\{s_1\}, \dots, \{s_n\}, p_1, \dots, p_n)$   
Apply (i).

# Order Independence

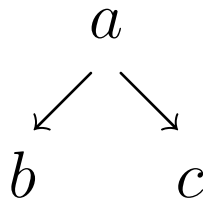
## Strict Dominance Theorem

All iterated eliminations of strictly dominated strategies yield the same outcome.

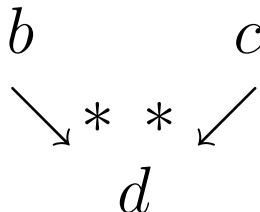
Crucial tool: **Newman's Lemma** (1942).

## Weak Confluence

- $A$  a set,  $\rightarrow$  a binary relation on  $A$ .  
 $\rightarrow^*$  : the transitive reflexive closure of  $\rightarrow$ .
- $b$  is a  $\rightarrow$ -**normal form of**  $a$  if
  - $a \rightarrow^* b$ ,
  - no  $c$  exists such that  $b \rightarrow c$ .
- If each  $a \in A$  has a unique normal form, then  $(A, \rightarrow)$  satisfies the **unique normal form property**.
- $\rightarrow$  is **weakly confluent** if  $\forall a, b, c \in A$



implies that for some  $d \in A$



## Newman's Lemma ('42)

Consider  $(A, \rightarrow)$  such that

- **no infinite**  $\rightarrow$  sequences exist,
- $\rightarrow$  is **weakly confluent**.

Then  $\rightarrow$  satisfies the **unique normal form property**.

## Application to Strict Dominance

### Observe:

- no infinite  $\rightarrow_S$  sequences exist.
- One can show that  $\rightarrow_S$  is weakly confluent.
- Conclusion: strict dominance is **order independent**.

## Strict Dominance: Summary

- Elimination of strictly dominated strategies preserves Nash equilibria.
- An iterated elimination of strictly dominated strategies yields a unique outcome.

## Weak Dominance: Recap

Consider a game  $(S_1, \dots, S_n, p_1, \dots, p_n)$ .

- A strategy  $s'_i$  **weakly dominates** a strategy  $s''_i$ , or equivalently, a strategy  $s''_i$  is **weakly dominated** by a strategy  $s'_i$  if

$$p_i(s'_i, s_{-i}) \geq p_i(s''_i, s_{-i})$$

for all  $s_{-i} \in S_{-i}$ , and

$$p_i(s'_i, s_{-i}) > p_i(s''_i, s_{-i})$$

for some  $s_{-i} \in S_{-i}$ .

- A strategy of player  $i$  is **weakly dominant** if it weakly dominates any other of his strategy.

## Example

Consider

		Two		
		Head	Tail	Edge
One	Head	-1, 1	1, -1	-1, -1
	Tail	1, -1	-1, 1	-1, -1
	Edge	-1, -1	-1, -1	-1, -1

- No strategy is **strictly dominated** by another one. So the IES yields no change.
- (Edge, Edge) is its only Nash equilibrium,
- For each player **Edge** is the only strategy that is **weakly dominated**.
- **Any** form of elimination of the **Edge** strategies yields the Matching Pennies game that has no Nash equilibrium.

So during this eliminating process we **'lost'** the only Nash equilibrium.

## Partial Result

Define  $G \rightarrow_W G'$  analogously as  $G \rightarrow_S G'$ .

### Weak Elimination Lemma

Suppose that  $G \rightarrow_W G'$ . If  $s$  is a Nash equilibrium of  $G'$ , then it is a Nash equilibrium of  $G$ .

### Weak Dominance Theorem

Suppose that  $G'$  is an outcome of an iterated elimination of weakly dominated strategies from the game  $G$ .

- (i) If  $s$  is a Nash equilibrium of  $G'$ , then it is a Nash equilibrium of  $G$ .
- (ii) If  $G$  is solved by an iterated elimination of weakly dominated strategies, then the resulting joint strategy is a Nash equilibrium of  $G$ .



# Problems with Order Independence

Consider

		Two	
		L	R
One	T	3,2	2,2
	M	1,1	0,0
	B	0,0	1,1

1. Eliminate B first:

		Two	
		L	R
One	T	3,2	2,2
	M	1,1	0,0

Now L weakly dominates R and T strictly dominates M, so we get:

		Two
		L
One	T	3,2

## Problems with Order Independence, ctd

2. Eliminate first M:

		Two	
		L	R
One	T	3, 2	2, 2
	B	0, 0	1, 1

Now R weakly dominates L and T strictly dominates B:

		Two	
		R	
One	T	2, 2	

3. Eliminate first both M and B:

		Two	
		L	R
One	T	3, 2	2, 2

So **three** different outcomes were produced.

## Weak Dominance: Summary

- Elimination of weakly dominated strategies can lead to a deletion of Nash equilibria.
- An iterated elimination of weakly dominated strategies does not yield a unique outcome.