CS 6280 - Multi agent systems

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Elimination of Dominated Strategies

Strict Dominance: Recap

Consider a game $(S_1, \ldots, S_n, p_1, \ldots, p_n)$.

- A strategy s_i' **strictly dominates** a strategy s_i'', or equivalently, a strategy s_i'' is **strictly dominated** by a strategy s_i' if

$$p_i(s_i', s_{-i}) > p_i(s_i'', s_{-i})$$

  for all $s_{-i} \in S_{-i}$.

- A strategy of player i is **strictly dominant** if it strictly dominates any other of his strategy.

Example

Consider

		Two		
		L	M	R
One	T	3,3	3,1	3,2
	B	2,2	2,4	2,1

By eliminating all strictly dominated strategies the game is reduced to

		Two	
		L	M
One	T	3,3	3,1

Now, strategy M *is* strictly dominated by the strategy L. Eliminating it we obtain

		Two
		L
One	T	3,3

Conclusion

Rational players `One` and `Two` will play (`T`,`L`).

- Why? Common Knowledge of rational behaviour:

  `One` knows that `Two` will not play `R`.

  `Two` knows that `One` will not play `B`.

  `One` knows that `Two` knows that `One` will not play `B`. So `One` knows that `Two` knows that `One` will play `T`.

  ...

- How general is this elimination process?
- In the Battle of the Sexes game no strategy (strictly or weakly) dominates another:

		Woman F	Woman B
Man	F	2,1	0,0
	B	0,0	1,2

- Do we keep all Nash equilibria?

Iterated Deletion

- Given a game $G := (S_1, \ldots, S_n, p_1, \ldots, p_n)$ and non-empty sets of strategies $S'_1, \ldots, S'_n$ such that $S'_i \subseteq S_i$ for $i \in [1..n]$ we say that

$$G' := (S'_1, \ldots, S'_n, p_1, \ldots, p_n)$$

  is a **subgame** of G and identify in the context of G' each payoff function p_i with its restriction.

- Consider a game $G := (S_1, \ldots, S_n, p_1, \ldots, p_n)$ and its subgame $G' := (S'_1, \ldots, S'_n, p_1, \ldots, p_n)$. Let

$$G \rightarrow_S G'$$

  when $G \neq G'$ and for all $i \in [1..n]$
  each $s''_i \in S_i \setminus S'_i$ is strictly dominated in G by some $s'_i \in S_i$.

Note: we do not require that all strictly dominated strategies are deleted.

Iterated Deletion and Nash Equilibria

Strict Elimination Lemma

Suppose that $G \rightarrow_S G'$. Then s is a Nash equilibrium of G' iff it is a Nash equilibrium of G.

Proof. Let

$$G := (S_1, \ldots, S_n, p_1, \ldots, p_n),$$

and

$$G' := (S'_1, \ldots, S'_n, p_1, \ldots, p_n).$$

($\Rightarrow$) Suppose s is not a Nash equilibrium of G. Then for some $i \in [1..n]$ and $s'_i \in S_i$

$$p_i(s'_i, s_{-i}) > p_i(s).$$

Choose s'_i for which $p_i(s'_i, s_{-i})$ attains the maximum. s'_i is eliminated since s is a Nash equilibrium of G'. So for some $s^*_i \in S_i$

$$p_i(s^*_i, s''_{-i}) > p_i(s'_i, s''_{-i}) \text{ for all } s''_{-i} \in S_{-i}.$$

In particular

$$p_i(s^*_i, s_{-i}) > p_i(s'_i, s_{-i}),$$

which contradicts the choice of s'_i.

Iterated Deletion and Nash Equilibria, ctd

($\Leftarrow$) For each player the set of his strategies in G' is a subset of the set of his strategies in G.

So to prove that s is a Nash equilibrium of G' it suffices to prove that no strategy constituting s is eliminated.

Suppose otherwise. Then some s_i is eliminated, so for some $s'_i \in S_i$

$$p_i(s'_i, s''_{-i}) > p_i(s_i, s''_{-i}) \text{ for all } s''_{-i} \in S_{-i}.$$

In particular

$$p_i(s'_i, s_{-i}) > p_i(s_i, s_{-i}),$$

so s is not a Nash equilibrium of

$$(S_1, \ldots, S_n, p_1, \ldots, p_n).$$

Iterated Deletion, ctd

- G' is **an outcome of an iterated elimination of strictly dominated strategies from the game G (IES)** if for no game G'', $G' \rightarrow_S G''$.
- **G is solved by an iterated elimination of strictly dominated strategies** if in G' each player has just one strategy.

Theorem

Suppose that G' is an outcome of an IES starting in the game G.

(i) Then s is a Nash equilibrium of G' iff it is a Nash equilibrium of G.

(ii) If G is solved by an iterated elimination of strictly dominated strategies, then the resulting joint strategy is a unique Nash equilibrium of G.

Iterated Deletion, ctd

In other words,

- each Nash equilibrium of the initial game **survives** any iterated elimination of strictly dominated strategies,
- each Nash equilibrium of an outcome of an iterated elimination of strictly dominated strategies **is also** a Nash equilibrium of the initial game,
- if a game is solved by an iterated elimination of strictly dominated strategies, then the **resulting joint strategy** is its Nash equilibrium.

Proof.

(i) By the repeated application of the Strict Elimination Lemma.

(ii) Note that $(s_1, \ldots s_n)$ is a unique Nash equilibrium of the game $(\{s_1\}, \ldots, \{s_n\}, p_1, \ldots, p_n)$ Apply (i).

Order Independence

Strict Dominance Theorem

All iterated eliminations of strictly dominated strategies yield the same outcome.

Crucial tool: **Newman's Lemma** (1942).

Weak Confluence

- A a set, $\longrightarrow$ a binary relation on A.
  $\longrightarrow^*$: the transitive reflexive closure of $\longrightarrow$.
- b is a $\longrightarrow$-**normal form of** a if
  - $a \longrightarrow^* b$,
  - no c exists such that $b \longrightarrow c$.
- If each $a \in A$ has a unique normal form, then $(A, \longrightarrow)$ satisfies the **unique normal form property**.
- $\longrightarrow$ is **weakly confluent** if $\forall a, b, c \in A$

$$
\begin{array}{ccc}
 & a & \\
\swarrow & & \searrow \\
b & & c
\end{array}
$$

implies that for some $d \in A$

$$
\begin{array}{ccc}
b & & c \\
\searrow^{*} & & {}^{*}\swarrow \\
 & d &
\end{array}
$$

Newman's Lemma ('42)

Consider $(A, \rightarrow)$ such that

- **no infinite** $\rightarrow$ sequences exist,
- $\rightarrow$ is **weakly confluent**.

Then $\rightarrow$ satisfies the **unique normal form property**.

Application to Strict Dominance

Observe:

- no infinite $\rightarrow_S$ sequences exist.
- One can show that $\rightarrow_S$ is weakly confluent.
- Conclusion: strict dominance is **order independent**.

Strict Dominance: Summary

- Elimination of strictly dominated strategies preserves Nash equilibria.
- An iterated elimination of strictly dominated strategies yields a unique outcome.

Weak Dominance: Recap

Consider a game $(S_1, \ldots, S_n, p_1, \ldots, p_n)$.

- A strategy s_i' **weakly dominates** a strategy s_i'', or equivalently, a strategy s_i'' is **weakly dominated** by a strategy s_i' if

  $$p_i(s_i', s_{-i}) \geq p_i(s_i'', s_{-i})$$

  for all $s_{-i} \in S_{-i}$, and

  $$p_i(s_i', s_{-i}) > p_i(s_i'', s_{-i})$$

  for some $s_{-i} \in S_{-i}$.

- A strategy of player i is **weakly dominant** if it weakly dominates any other of his strategy.

Example

Consider

		Two		
		Head	Tail	Edge
	Head	-1,1	1,-1	-1,-1
One	Tail	1,-1	-1,1	-1,-1
	Edge	-1,-1	-1,-1	-1,-1

- No strategy is **strictly dominated** by another one. So the IES yields no change.
- (`Edge`, `Edge`) is its only Nash equilibrium,
- For each player `Edge` is the only strategy that is **weakly dominated**.
- **Any** form of elimination of the `Edge` strategies yields the Matching Pennies game that has no Nash equilibrium.

So during this eliminating process we **'lost'** the only Nash equilibrium.

Partial Result

Define $G \rightarrow_W G'$ analogously as $G \rightarrow_S G'$.

Weak Elimination Lemma

Suppose that $G \rightarrow_W G'$. If s is a Nash equilibrium of G', then it is a Nash equilibrium of G.

Weak Dominance Theorem

Suppose that G' is an outcome of an iterated elimination of weakly dominated strategies from the game G.

(i) If s is a Nash equilibrium of G', then it is a Nash equilibrium of G.

(ii) If G is solved by an iterated elimination of weakly dominated strategies, then the resulting joint strategy is a Nash equilibrium of G.

Problems with Order Independence

Consider

		Two	
		L	R
	T	3,2	2,2
One	M	1,1	0,0
	B	0,0	1,1

1. Eliminate B first:

		Two	
		L	R
One	T	3,2	2,2
	M	1,1	0,0

Now L weakly dominates R and T strictly dominates M, so we get:

		Two
		L
One	T	3,2

Problems with Order Independence, ctd

2. Eliminate first `M`:

		Two L	Two R
One	T	3,2	2,2
	B	0,0	1,1

Now `R` weakly dominates `L` and `T` strictly dominates `B`:

		Two R
One	T	2,2

3. Eliminate first both `M` and `B`:

		Two L	Two R
One	T	3,2	2,2

So **three** different outcomes were produced.

Weak Dominance: Summary

- Elimination of weakly dominated strategies can lead to a deletion of Nash equilibria.
- An iterated elimination of weakly dominated strategies does not yield a unique outcome.