Problem Set 1: Part A
Caput Logic, Language and Information: Social Software

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Note that the homework comes in two parts — the first part is about decision theory and the second part (to be added next week) will cover some basics of game theory.

[M] is *Game Theory* by R. Myerson.

1. A preference relation $\prec$ is **rational** if for each x, y, z if $x \prec y$ and $z \not\prec y$ then $x \prec z$. Let $X \subseteq \mathbb{R} \times \mathbb{R}$ and suppose that $\prec$ is the lexiographic ordering. That is for $r, s \in X$ (with $r = (r_1, r_2)$ and $s = (s_1, s_2)$) we write $r \prec s$ iff either $r_1 < s_1$ or $r_1 = s_1$ and $r_2 < s_2$. Prove that $\prec$ is rational.

2. We presented a number of axioms about a decision-makers preferences over lotteries. For each of the following axioms write a short paragraph that either justifies the axiom or gives an argument against the axiom. (Note: the answer that they are needed to prove the expected utility theorem will **not** be accepted!). Let f, g be lotteries and S a set of states.

   (a) *(Completeness)* $f \succeq_S g$ or $g \succeq_S f$.

   (b) *(Transitivity)* If $f \succeq_S g$ and $g \succeq_S h$ then $f \succeq_S h$.

   (c) *(Relevance)* If, for all $t \in \Omega$, $f(\cdot|t) = g(\cdot|t)$ then $f \sim_S g$.

   (d) *(Monotonicity)* If $f \succ_S g$ and $0 \leq \beta < \alpha \leq 1$ then $\alpha f + (1-\alpha)g \succ_S \beta f + (1-\beta)g$.

   (e) *(Continuity)* If $f \succeq_S g$ and $g \succeq_S h$ then there exists a number γ such that $0 \leq \gamma \leq 1$ and $g \sim_S \gamma f + (1-\gamma)h$.

   6. *(Objective substitution)* If $e \succeq_S f$, $h \succeq_S g$ and $0 \leq \alpha \leq 1$, then $\alpha e + (1-\alpha)h \succeq_S \alpha f + (1-\alpha)g$.

   7. *(Subjective substitution)* If $g \succeq_S f$, $g \succeq_T f$ and $S \cap T = \emptyset$ then $g \succeq_{S \cup T} f$.

3. (Exercise 1.1 in [M]) Suppose that the set of prizes X is a finite subset of $\mathbb{R}$ and a prize $x \in X$ denotes an award of x euros. A decison-maker says

that, if she knew that the true state of the worlds was in some set T, then she would weakly prefer a lottery f over another lottery g (i.e., $f \succeq_T g$) iff

$$\min_{s \in T} \sum_{x \in X} x f(x|s) \geq \min_{s \in T} \sum_{x \in X} x g(x|s)$$

That is, she prefers the lottery that gives the higher expected payoff in the worst possible state. Which of the axioms (if any) presented in class does this preference relation violate? Be sure to explain your answers (give a proof or provide a counterexample).

4. (Exercise 1.4 in [M]) Consider the following two axioms discussed in class:

   (a) *(Completeness)* $f \succeq_S g$ or $g \succeq_S f$.

   (b) *(Strict Objective substitution)* If $e \succ_S f$, $h \succeq_S g$ and $0 < \alpha \leq 1$, then $\alpha e + (1-\alpha)h \succ_S \alpha f + (1-\alpha)g$.

   Prove that these two axioms together imply both of the following

   (a) *(Transitivity)* If $f \succeq_S g$ and $g \succeq_S h$ then $f \succeq_S h$.

   (b) *(Monotonicity)* If $f \succ_S g$ and $0 \leq \beta < \alpha \leq 1$ then $\alpha f + (1-\alpha)g \succ_S \beta f + (1-\beta)g$.