Problem Set 2
Caput Logic, Language and Information: Social Software

Institute for Logic, Language and Computation
Universiteit van Amsterdam

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These questions are from *Axioms of Cooperative Decision Making* by H. Moulin.

1. Suppose there are n different candidates and m agents. Consider the following voting procedure.

   Borda Rule: Each agent ranks the candidates from top to bottom (ties are not allowed). Given a ranking of the candidate, each candidate receives $n-1$ points if he is at the top of the ranking, $n-2$ points if he is ranked second, ..., 0 points if he is ranked last. The candidates Borda score is the total number of points the agent receives. The Borda winner is the agent with the largest Borda score.

   For example, suppose there are 15 agents and 3 candidates (say a, b, and c). Say that agents report the following rankings

No. of voters:	5	6	4
	a	c	b
	c	a	a
	b	b	c

   Then a's Borda score is 20 $(5 \cdot 2 + 6 \cdot 1 + 4 \cdot 1)$, b's Borda score is 8 $(5 \cdot 0 + 6 \cdot 0 + 4 \cdot 2)$ and c's Borda score is 17 $(5 \cdot 1 + 6 \cdot 2 + 4 \cdot 0)$. So, a is the Borda winner. Show that the Borda score does not satisfy independance of irrelevant alternatives[1].

   Hint: Suppose there are 4 candidates (a, b, c and d) and 7 voters with the following rankings:

No. of voters:	3	2	2
	c	b	a
	b	a	d
	a	d	c
	d	c	b

   Verify that the Borda ranking is $a > b > c > d$. Remove the loser d and recompute the Borda rankings among the remaining individuals.

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[1]This was first observed by P. Fishburn in 1984.

2. Let $\mathcal{A}$ be a set of agents and X a set of candidates. We have seen that the simple majority rule is not transitive. However, suppose we fix and ordering of the agents $\mathcal{A} = \{1, 2, \ldots, n\}$ and consider only profiles $\vec{\mathcal{P}}$ (individual prefrences are assumed to be complete, transitive and asymmetric) that satisfy the following property

   For all $i, j, k \in \mathcal{A}$ and all $a, b \in X$, if $i < k < j$ and aP_ib, aP_jb then aP_kb.

   Prove that the strict majority rule is transitive if we restrict the domain of profiles to only those satisfying the above property.

3. **A variant of Arrow's Theorem** An acyclic relation is a complete binary relation (R) whose strict component (P) has no cycles (i.e., no sequence $a_0, \ldots, a_n$ such that a_iPa_{i+1} for $i = 0, \ldots, n-1$ and a_nPa_0). A social welfare relation R is strictly monotonic if given two profiles $\vec{\mathcal{P}}$ and $\vec{\mathcal{Q}}$ the only chagne from $\vec{\mathcal{P}}$ to $\vec{\mathcal{Q}}$ is to improve the relative position of a and $\vec{\mathcal{P}} \neq \vec{\mathcal{Q}}$ then $a\mathsf{R}(\vec{\mathcal{P}})b$ then $a\mathsf{R}(\vec{\mathcal{Q}})b$. Prove that if there are at least three candidates, or outcomes, and R satisfies IIA and maps profiles to acyclic relations, then R is dictatorial.