Lecture 10: Strategy Logics

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Caput Logic, Language and Information: Social Software

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Introduction

Imagine a set of agents (players, system components) taking actions, simultaneously or in turns, on a common set of states — and thus effecting transitions between these states.

Assume the agents pursue certain goals and in that pursuit they can form **coalitions**

The objective is to develop formal tools for reasoning about coalitions of agents and their ability to achieve specified outcomes in such situations.

(Temporal) Logics for Reasoning about Actions

- *Linear Time Temporal Logic* (**LTL**) [Pnuelli, 1977]: Reasoning about computations:

  $\Diamond\phi$: ϕ is true some time in the future.

- *Branching-time Temporal Logic* (**CTL**, **CTL***) [Clarke and Emerson, 1981, Emerson and Halpern, 1986]: Allows quantification over paths:

  $\exists\Diamond\phi$: there is a path in which ϕ is eventually true.

- *Alternating-time Temporal Logic* (**ATL**, **ATL***) [Alur, Henzinger, Kupferman, 1997]: Selective quantification over paths:

  $\langle\langle A\rangle\rangle \bigcirc \phi$: The coalition A has a joint strategy to ensure that ϕ is true at the next moment.

ATL:Syntax

Let Σ be a set of agents, Π a set of propositional variables and $A \subseteq \Sigma$.

1. p where $p \in \Pi$
2. $\neg\phi$
3. $\phi \vee \psi$
4. $\langle\langle A \rangle\rangle \bigcirc \phi$ meaning 'The coalition A can force in the next move an outcome satisfying ϕ'
5. $\langle\langle A \rangle\rangle \Box \phi$ meaning 'The coalition A can maintain forever outcomes satisfying ϕ'
6. $\langle\langle A \rangle\rangle \phi U \psi$ meaning 'The coalition A can eventually force an outcome satisfying ψ while meanwhile maintaining the truth of ϕ

Pauly's Coalitional Logic: Syntax

Given a finite non-empty set of agents N and a set of atomic propositions Φ_0, a formula ϕ can have the following syntactic form

$$\phi \ := \ \bot \mid p \mid \neg\phi \mid \phi \vee \phi \mid [C]\phi$$

where $p \in \Phi_0$ and $C \subseteq N$.

$[C]\phi$ is intended to mean "coalition C can (locally) force ϕ to be true"

M. Pauly. *Logics for Social Software*. Ph.D. Thesis, ILLC.

Coalitional Logic: Multi-player Game Models

A **Strategic Game Form** is a tuple $\langle N, \{\Sigma_i \mid i \in N\}, Q, o \langle$ where

- N is a set of agents
- Σ_i is a set of actions
- Q is a set of states
- $o : \Pi_{i \in N} \Sigma_i \rightarrow Q$ assigns an outcome to each choice of aciton.

Let Γ_Q^N be the set of all such strategic game forms.

A **Multi-Player Game Model** is a tuple $\langle Q, \gamma, \pi \rangle$ where Q is a set of states and $\gamma : Q \rightarrow \Gamma_Q^N$ associates strategic games form to each state

$$q \models [C]\phi \text{iff } \exists\, \sigma_C \;\forall\, \sigma_{N-C},\; o(\sigma_C, \sigma_{N-C}) \models \phi$$

Effectivity Functions

Let G be a strategic game.

$$X \in E_G^\alpha(C) \text{iff } \exists \sigma_C \forall \sigma_{\overline{C}} \;\; o(\sigma_C, \sigma_{\overline{C}}) \in X$$

$$X \in E_G^\beta(C) \text{iff } \forall \sigma_{\overline{C}} \exists \sigma_C \;\; o(\sigma_C, \sigma_{\overline{C}}) \in X$$

$E_G^\alpha \subseteq E_G^\beta$

$$E_G^\beta \not\subseteq E_G^\alpha$$

Player 1 chooses the row, Player 2 chooses the column, Player 3 chooses the table

	l	m	r
l	s_1	s_2	s_1
r	s_2	s_1	s_3

	l	m	r
l	s_3	s_1	s_2
r	s_2	s_3	s_3

$$\{s_2\} \in E_G^\beta(\{2\}) \text{ but } \{s_2\} \notin E_G^\beta(\{2\})$$

Coalitional Logic: Coalition Effectivity Models

An effectivity function is **playable** iff

1. For each $C \subseteq N$, $\varnothing \notin E(C)$
2. For each $C \subseteq N$, $Q \in E(C)$
3. If $X \notin E(N)$, then $Q - X \in \mathsf{E}(\varnothing)$
4. If $X \subseteq Y$ and $X \in E(C)$ then $Y \in E(C)$
5. for all $C_1, C_2 \subseteq N$ and $X_1, X_2 \subseteq Q$, if $C_1 \cap C_2 = \varnothing$, $X_1 \in E(C_1)$ and $X_2 \in E(C_2)$ then $X_1 \cap X_2 \in E(C_1 \cup C_2)$

Characterization Theorem: An α-effectivity function E is playable iff it is the effectivity function of some strategic game.

M. Pauly. *Logics for Social Software.* Ph.D. Thesis, ILLC.

Coalitional Logic: Coalition Effectivity Models

A **coalitional effectivity model** is a tuple $\langle Q, E, \pi \rangle$ where $E : Q \rightarrow (2^N \rightarrow 2^{2^Q})$ assigns a playable effectivity function to each state and π is a valuation function.

$$q \models [C]\phi \text{ iff } \|\phi\| \in E_q(C)$$

Coalitional Logic: Axiomatization

$(\bot)\quad \neg[C]\bot$

$(\top)\quad [C]\top$

$(N)\quad (\neg[\varnothing]\neg\phi \rightarrow [N]\phi)$

$(M)\quad [C](\phi \wedge \psi) \rightarrow [C]\psi$

$(S)\quad ([C_1]\phi_1 \wedge [C_2]\phi_2) \rightarrow [C_1 \cup C_2](\phi_1 \wedge \phi_2)$

provided $C_1 \cap C_2 = \varnothing$

$$\frac{\phi \qquad \phi \rightarrow \psi}{\psi}$$

$$\frac{\phi \leftrightarrow \psi}{[C]\phi \leftrightarrow [C]\psi}$$

Theorem Coalitional Logic is sound and strongly complete with respect to the class of effectivity models.

Theorem The complexity of the satisfiability problem of coalitional logic is PSPACE-complete.

M. Pauly. *A Modal Logic for Coalitional Powers in Games*. Journal of Logic and Computation (2002).

Coalition Logic is a Fragment of ATL

Define $[A]\phi$ to be $\langle\langle A\rangle\rangle \bigcirc \phi$

Multi-player Game Models and Concurrent-game Models only differ in notation

Coalitional Effectivity Models can be used as a semantics for **ATL**

Goranko and Jamroga. *Comparing Semantics of Logics fro Multi-Agent Systems*. See the website.

ATL, Semantics I (Concurrent Game Structure)

A **Concurrent Game Structure** is a tuple $\mathcal{S} = \langle n, Q, \Pi, \pi, d, \delta \rangle$ with

- A natural number $n \geq 1$ of players: $\Sigma = \{1, \ldots, n\}$
- A set Q of states
- A finite set Π of atomic propositions
- For each $q \in Q$, a set $\pi(q) \subseteq \Pi$ of atomic propositions true at q.

ATL, Semantics I (Concurrent Game Structure)

A **Concurrent Game Structure** is a tuple $\mathcal{S} = \langle n, Q, \Pi, \pi, d, \delta \rangle$ with

- For each $a \in \Sigma$ and each $q \in Q$, a natural number $d_a(q) \geq 1$ of moves available at state q to player a. Moves of player a at q are identified with numbers $1, \ldots, d_a(q)$.

  A **move vector** at q is a tuple $\langle j_1, \ldots, j_k \rangle$ such that $1 \leq j_a \leq d_a(q)$ for each player a.

  Write $D(q) = \{1, \ldots, d_1(q)\} \times \cdots \times \{1, \ldots, d_n(q)\} \subseteq \mathbb{N}^n$.

- For each $q \in Q$ and each move vector $\langle j_1, \ldots, j_n \rangle \in D(q)$, a state $\delta(q, j_1, \ldots, j_n) \in Q$ that results from the state q if each player $a \in \{1, \ldots, n\}$ chooses move j_a. δ is called the **transition function**.

Notation

- q' is a **successor** of q iff $\exists\langle j_1, \ldots, j_n\rangle \in D(q)$ such that $q' = \delta(q, j_1, \ldots, j_n)$

- A **computation** is an infinite sequence $\lambda = q_0 q_1 \ldots$ of states such that for all $i \geq 0$, q_{i+1} is a successor of q_i

- If a computation $\lambda = q_0 q_1 q_2 \cdots$ starts at q (i.e., $q_0 = q$), λ is called a q-computation

- Given a computation λ, $\lambda[i]$, $\lambda[0, i]$, and $\lambda[i, \infty]$ denote respectively the ith state of λ, the finite prefix $q_0 q_1 \cdots q_i$ and the infinite suffix $q_i q_{i+1} \cdots$

Strategies I

A **strategy** for player $a \in \Sigma$ is a function $f_a : Q^+ \to \mathbb{N}$ that maps every nonempty finite state sequence λ to a natural number such that if the last state is q, $1 \leq f_a(\lambda) \leq d_a(q)$.

For a state $q \in Q$ and $A \subseteq \Sigma$ an A-move is a tuple $(\sigma_a)_{a \in A}$ such that $1 \leq \sigma_a \leq d_a(q)$. Let $D_A(q)$ denote the set of A-moves.

A state q' is **consistent** with an A-move $\sigma \in D_A(q)$ when there is a move vector $\langle j_1, \ldots, j_n \rangle \in D(q)$ such that

1. $j_a = \sigma_a$ for all $a \in A$
2. $q' = \delta(q, j_1, \ldots, j_n)$

Strategies II

Let out(σ) denote the set of state consistent with σ

For $A \subseteq \Sigma$ an A-strategy is a mapping $F_A : Q^+ \rightarrow \cup\{D_A(q) \mid q \in Q\}$ such that for all $\lambda \in Q^*$, and for all $q \in Q$, $F_A(\lambda \cdot q) \in D_A(q)$.

A q-computation $\lambda = q_0 q_1 \ldots$ with $q_0 = q$ is **consistent** with F_A written $\lambda \in out(q, F_A)$ if for all $i \geq 0$, q_{i+1} is a successor of q_i and $q_{i+1} \in out(F_A(\lambda[0, i]))$.

Definition of Truth

- For $p \in \Pi$, $q \models p$ iff $p \in \pi(q)$
- $q \models \neg\phi$ iff $q \not\models \phi$
- $q \models \phi \vee \psi$ iff $q \models \phi$ of $q \models \psi$
- $q \models \langle\!\langle A \rangle\!\rangle \bigcirc \phi$ iff there exists an A-strategy F_A such that for each computation $\lambda \in out(q, F_A)$ we have $\lambda[1] \models \phi$; equivalently, iff there exists an A-move $\sigma \in D_A(q)$ such that for all $q' \in out(\sigma)$, $q' \models \phi$.
- $q \models \langle\!\langle A \rangle\!\rangle \Box \phi$ iff there exists an A-strategy F_A such that for each computation $\lambda \in out(q, F_A)$ and all $i \geq 0$, we have $\lambda[i] \models \phi$
- $q \models \langle\!\langle A \rangle\!\rangle \phi U \psi$ iff there exists an A-strategy F_A such that for each computation $\lambda \in out(q, F_A)$ there exists a position $i \geq 0$ such that $\lambda[i] \models \psi$ and for all $0 \leq j < i$ we have $\lambda[j] \models \phi$

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Alternative Semantics for ATL: Alternating Transition Systems

An **alternating transition system** is a tuple $\langle \Pi, N, Q, \pi, \delta \rangle$ where

- Π is a set of atomic propositions
- N is a set of agents (nonempty and finite)
- Q is a set of states (nonempty)
- $\pi : Q \to 2^{\Pi}$ is a valuation function
- $\delta : Q \times N \to 2^{2^{Q}}$ is a **transition function**. For each $i \in N$, $Q_i \in \delta(q, i)$ means i can force the state to be in Q_i. We assume that for each that the system is deterministic: given a state q and the agents choices in q, $Q_1, \ldots, Q_n$, $Q_1 \cap \cdots Q_n$ is a singleton.

Axiomatization

Coalitional Logic Axioms:

(TAUT) Enough propositional tautologies

($\bot$) $\neg\langle\langle A\rangle\rangle \bigcirc \bot$

($\top$) $\langle\langle A\rangle\rangle \bigcirc \top$

(Σ) $\neg\langle\langle \emptyset\rangle\rangle \bigcirc \neg\phi \rightarrow \langle\langle \Sigma\rangle\rangle \bigcirc \phi$

(S) $\langle\langle A_1\rangle\rangle \bigcirc \phi_1 \wedge \langle\langle A_2\rangle\rangle \bigcirc \phi_2 \rightarrow \langle\langle A_1 \cup A_2\rangle\rangle \bigcirc (\phi_1 \wedge \phi_2)$

Axiomatization

Fixed Point Axioms:

(FP$_\Box$) $\langle\!\langle A\rangle\!\rangle\Box\phi \leftrightarrow \phi \wedge \langle\!\langle A\rangle\!\rangle \bigcirc \langle\!\langle A\rangle\!\rangle\Box\phi$

(GFP$_\Box$) $\langle\!\langle \varnothing\rangle\!\rangle\Box(\theta \rightarrow (\phi \wedge \langle\!\langle A\rangle\!\rangle \bigcirc \theta)) \rightarrow \langle\!\langle \varnothing\rangle\!\rangle\Box(\theta \rightarrow \langle\!\langle A\rangle\!\rangle\Box\phi)$

(FP$_U$) $\langle\!\langle A\rangle\!\rangle\phi U\psi \leftrightarrow \psi \vee (\phi \wedge \langle\!\langle A\rangle\!\rangle \bigcirc \langle\!\langle A\rangle\!\rangle\phi U\psi)$

(LFP$_U$) $\langle\!\langle \varnothing\rangle\!\rangle\Box((\psi \vee (\phi \wedge \langle\!\langle A\rangle\!\rangle \bigcirc \theta)) \rightarrow \theta) \rightarrow \langle\!\langle \varnothing\rangle\!\rangle\Box(\langle\!\langle A\rangle\!\rangle\phi U\psi \rightarrow \theta)$

Axiomatization

Rules:

(Modus Ponens) $$\frac{\phi \qquad \phi \rightarrow \psi}{\psi}$$

($\langle\langle A\rangle\rangle\bigcirc$-Monotonicity) $$\frac{\phi \rightarrow \psi}{\langle\langle A\rangle\rangle \bigcirc \phi \rightarrow \langle\langle A\rangle\rangle \bigcirc \psi}$$

($\langle\langle A\rangle\rangle\Box$-Necessitation) $$\frac{\phi}{\langle\langle A\rangle\rangle\Box\phi}$$

Results

Theorem All of the semantics (concurrent game structures, alternating transitions systems and coalitional effectivity models) are equivalent.

Goranko and Jamroga. *Comparing Semantics of Logics for Multi-Agent Systems*. See the website.

Theorem **ATL** is sound and (weakly) complete.

Theorem Given a finite set of players, the satisfiability problem for **ATL**-formulas over N with respect to concurrent game structures over N is EXPTIME-complete.

Goranko and van Drimmelen. *Complete Axiomatization and Decidability of the Alternating-Time Temporal Logic*. Theoretical Computer Science (2005).

ATEL

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Some issues

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- By adding knowledge operators, we can express interesting properties:

$$\langle\langle \{1,2\} \rangle\rangle (\neg K_3 p \; U \; K_2 p)$$

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- By adding knowledge operators, we can express interesting properties:

$$\langle\langle \{1,2\} \rangle\rangle (\neg K_3 p \; U \; K_2 p)$$

- *de re* vs. *de dicto* knowledge of strategies:
  - agent i knows there is a winning strategy to achieve ϕ

– there is a winning strategy which i knows will achieve ϕ

Where do the coalitions come from?

There is a lot of work on cooperative game theory which attempts to describe *why* agents form a coalition.

One possible way is via aggregation procedures:

M. Pauly. *Axiomatising Judgement Aggregation Procedures in a Minimal Logical Language*. forthcoming in KRA.

M. Pauly. *On the Role of Language in Social Choice Theory*. unpublished.

First, A Paradox (Kornhauser and Sager 1993)

P: a valid contract was in place

Q: the defendant's behavior was such as to breach a contract of that kind

R: the court is required to find the defendant liable.

	P	Q	$(P \wedge Q) \leftrightarrow R$	R
1	yes	yes	yes	yes
2	yes	no	yes	no
3	no	yes	yes	no

Should we accept R?

	P	Q	$(P \wedge Q) \leftrightarrow R$	R
1	yes	yes	yes	yes
2	yes	no	yes	no
3	no	yes	yes	no

Should we accept R? No, a simple majority votes no.

	P	Q	$(P \wedge Q) \leftrightarrow R$	R
1	yes	yes	yes	yes
2	yes	no	yes	no
3	no	yes	yes	no

Should we accept R? Yes, a majority votes yes for P and Q and $(P \wedge Q) \leftrightarrow R$ is a legal doctrine.

	P	Q	$(P \wedge Q) \leftrightarrow R$	R
1	yes	yes	yes	yes
2	yes	no	yes	no
3	no	yes	yes	no

List and Pettit Impossibility Result

Suppose there are n agents and let $\mathcal{L}$ be a propositional language.

Personal judgement sets: a *consistent, complete* and *deductively closed* set of formulas — a maximally consistent set.

A **collective judgement aggregation function:** Let $\mathbb{M} = \{\Gamma \mid \Gamma \text{ is a maximally consistent set}\}$ then a collective aggregation function is defined as follows:

$$F : \mathbb{M}^n \to \mathbb{M}$$

Some Conditions

Universal Domain F is a total function

Anonymity For all $\vec{\Gamma} \in \mathbb{M}^n$, $F(\Gamma_1, \ldots, \Gamma_n) = F(\Gamma_{\pi(1)}, \ldots, \Gamma_{\pi(n)})$ for all permutations π

Systematicity There exists a function $f : \{0,1\}^n \rightarrow \{0,1\}$ such that for any $\vec{\Gamma} \in \mathbb{M}^n$,
$F(\Gamma_1, \ldots, \Gamma_n) = \{\phi \in X \mid f(\delta_1(\phi), \ldots, \delta_n(\phi)) = 1\}$, where, for each agent i and each $\phi \in X$, $\delta_i(\phi) = 1$ if $\phi \in \Gamma_i$ and $\delta_i(\phi) = 0$ if $\phi \notin \Gamma_i$

Theorem [List and Pettit, 2001] There exists no judgement aggregation function generating complete, consistent and deductively closed collective sets of judgements which satisfies Universal Domain, Anonymity and Systematicity.

Brief Survey of the Literature

- See `personal.lse.ac.uk/LIST/doctrinalparadox.htm` for a detailed overview of the current state of affairs. Some highlights:

- Other impossibility results: Pauly and van Hees (2003), van Hees (2004), GŁrdenfors (2004), among others

- List and Pettit, 2005 compare their impossibility result with Arrow's Theorem

A Logic for Aggregation Procedures

Let Φ_I be the set of **individual formulas** (standard propositional language)

V_I the set of individual valuations

Φ_C the set of **collective formulas** (standard modal language)

$\Box\phi$: *The group collectively accepts* ϕ.

Conclusion Driven Procedure: $\Box\phi \wedge \Box\psi$

Premise Driven Procedure: $\Box(\phi \wedge \psi)$

For majority voting, $\Box\phi \wedge \Box\psi \rightarrow \Box(\phi \wedge \psi)$ but *not conversely.*

Let $\Phi_{\Box I} = \{\Box\phi \mid \phi \in \Phi_I\}$, a collective valuation is a function $v : \Phi_{\Box I} \rightarrow \{0, 1\}$

A **decision method** is a function $D : \{0, 1\}^n \rightarrow \{0, 1\}$

Dictatorship: there is a $d \in N$ such that $D(x_1, \ldots, x_n) = x_d$ for all $x_1, \ldots, x_n \in D$

Consensus: $D(x_1, \ldots, x_n) = 1$ iff $x_i = 1$ for all i

Majority: $D(x_1, \ldots, x_n) = 1$ iff $|\{i \mid x_i = 1\}| > \frac{1}{2}|N|$

A model $v \in V_C$ is n-**systematic** iff there is some n-ary decision method D and there are n individual valuation $v_1, \ldots, v_n \in V_I$ such that for all $\phi \in \Phi_I$, $v(\Box\phi) = D(v_1(\phi), \ldots, v_n(\phi))$

Similarly for the other decision procedures

Connection with simple games.

$\mathsf{E} = \{\Box\phi \to \Box\psi \mid \phi \to \psi \in \Phi_I \text{ is a tautology}\}$

Theorem (Pauly, 2005) $V_C(\mathsf{E}) = \mathcal{SYS}_n$, provided $n \geq 2^{|\Phi_0|}$.

KD extends E with the following axiom schemes:

M. $\Box(\phi \wedge \psi) \rightarrow (\Box\phi \wedge \Box\psi)$

C. $(\Box\phi \wedge \Box\psi) \rightarrow (\Box\phi \wedge \Box\psi)$

N. $\Box\top$

D. $\neg\Box\bot$

Theorem (Pauly, 2005) $V_C(\mathsf{KD}) = \mathcal{CON}_n$, provided $n \geq 2^{|\Phi_0|}$.

MCY contains M, C and the following scheme:

Y. $\neg\Box\phi \leftrightarrow \Box\neg\phi$

Theorem (Pauly, 2005) $V_C(\mathsf{MCY}) = \mathcal{DIC}$.

STEM contains E, M and all instances of the following schemes

S. $[>]\phi \rightarrow \neg[>]\neg\phi$

T. $([\geq]\phi_1 \wedge \cdots \wedge [\geq]\phi_k \wedge [\leq]\psi_1 \wedge \cdots \wedge [\leq]\psi_k) \rightarrow \bigwedge_{1 \leq i \leq k}([=]\phi_i \wedge [=]\psi_i)$
where $\forall v \in V_I : |\{i \mid v_(\phi_i) = 1\}| = |\{i \mid v(\psi_i) = 1\}|$

Theorem (Pauly, 2005) $V_C(\mathsf{STEM}) = \mathcal{MAJ}$. (Nontrivial!!)

Next Week: Class Canceled, sort of — I will speak at the Segerberg Celebration