Lecture 2: Some Decision and Game Theory

Eric Pacuit

ILLC, University of Amsterdam

staff.science.uva.nl/~epacuit

epacuit@science.uva.nl

Lecture Date: February 16, 2006

Caput Logic, Language and Information: Social Software

staff.science.uva.nl/~epacuit/caputLLI.html

For the next few courses...

Goal: Discuss the basic tools needed to perform *Gedankenexperiments* about groups of individual in social situations under a variety of assumptions.

1. Part 1: Basic Decision Theory:
   - Preferences and utilities
   - Expected utility theorem
2. Part 2: Basic non-cooperative game theory:
   - Normal form games
   - Equilibrium concepts

Part 1: Bayesian decision theory

Part 1: Bayesian decision theory

Agents will choose an act of maximum estimated desirability.

Part 1: Bayesian decision theory

Agents will choose an act of maximum estimated desirability.

How do we represent an agent's "desirability"?

Part 1: Bayesian decision theory

Agents will choose an act of maximum estimated desirability.

How do we represent an agent's "desirability"?

See *The Logic of Decision* by R. Jeffrey.

Preferences and Utilities

Let X be a set of **outcomes**.

A **preference** is a subset of $X \times X$

For $x, y \in X$, $x \prec y$ intended to mean "y is **more desirable** than x"

A **utility** function is a function $u : X \to \mathbb{R}$ (or any ordered set).

Obviously a utility function generates a preference ordering.

Preferences and Utilities

Typical Assumptions: reflexive, irreflexive, symmetric, asymmetric, transitive, complete.

Preferences and Utilities

Typical Assumptions: reflexive, irreflexive, symmetric, asymmetric, transitive, complete.

Preferences and Utilities

Typical Assumptions: reflexive, irreflexive, symmetric, asymmetric, transitive, complete.

Preferences and Utilities

Typical Assumptions: reflexive, irreflexive, symmetric, asymmetric, transitive, complete.

Preferences and Utilities

Typical Assumptions: reflexive, irreflexive, symmetric, asymmetric, transitive, complete.

Preferences and Utilities

Typical Assumptions: reflexive, irreflexive, symmetric, asymmetric, transitive, complete.

Preferences and Utilities

Typical Assumptions: reflexive, irreflexive, symmetric, asymmetric, transitive, complete.

Preferences and Utilities

Typical Assumptions: reflexive, irreflexive, symmetric, asymmetric, transitive, complete.

Indifference: $x \sim y$ if $x \not\prec y$ and $y \not\prec x$.

Not Worse Than: $x \preceq y$ if $x \prec y$ or $x \sim y$

Preferences and Utilities

Rational Preference Relation: if $x_1 \prec x_2$ and $x_3 \not\prec x_2$, then $x_1 \prec x_3$.

Preferences and Utilities

Rational Preference Relation: if $x_1 \prec x_2$ and $x_3 \not\prec x_2$, then $x_1 \prec x_3$.

Lemma Let $\prec$ be a transitive relation on X. Then the indifference relation $\sim$ defined from $\prec$ is transitive iff $\prec$ is rational.

Proposition If $\prec$ is a rational preference on X, then

1. $\preceq$ is complete, reflexive and transitive, and vice versa
2. $\sim$ is an equivalence relation

Preferences and Utilities

A utility function **represents** the preference relation $\prec$ on A if $x_1 \prec x_2$ iff $u(x_1) < u(x_2)$

Proposition

1. If a preference relation $\prec$ is representable by a utility function, then it is a rational preference relation
2. If the set of alternatives is finite and $\prec$ is a rational preference, then there exits a utility function representing the preference $\prec$.

From Decision Theory to Game Theory

...to understand the fundamental ideas of game theory, one should begin by studying decision theory.

Game Theory: Analysis of Conflict by R. Myerson

From Decision Theory to Game Theory

...to understand the fundamental ideas of game theory, one should begin by studying decision theory.

Game Theory: Analysis of Conflict by R. Myerson

- Any rational decision-maker's behaviour should be describable by a utility function, which gives a quantitative characterization of his preference for outcomes and prizes and

From Decision Theory to Game Theory

...to understand the fundamental ideas of game theory, one should begin by studying decision theory.

Game Theory: Analysis of Conflict by R. Myerson

- Any rational decision-maker's behaviour should be describable by a utility function, which gives a quantitative characterization of his preference for outcomes and prizes and

- a subjective probability distribution, which characterizes his beliefs about all relevant unknown factors.

From Decision Theory to Game Theory

...to understand the fundamental ideas of game theory, one should begin by studying decision theory.

Game Theory: Analysis of Conflict by R. Myerson

- Any rational decision-maker's behaviour should be describable by a utility function, which gives a quantitative characterization of his preference for outcomes and prizes and
- a subjective probability distribution, which characterizes his beliefs about all relevant unknown factors.
- As new information becomes available to a decision maker, his probabilities should be revised according to Bayes's rule.

Expected Utility Theorem: Notation

- Let A be an arbitrary set, then $\Delta(A)$ is the set of a probability distributions on A.

- Let X be a *finite* set of **prizes** or **outcomes**.

- Let Ω be a *finite* set of **states**

- A **lottery** is a funciton $f : \Omega \to \Delta(X)$.

  The intended interpretation of $f(x|t)$ is "the *objective* probability of getting prize x given that the current state is t".

- For each $x \in X$, define $[x](y|t) = \begin{cases} 1 & \text{if } y = x \\ 0 & \text{otherwise} \end{cases}$

Expected Utility Theorem: Notation

- Let $\Sigma = \{S \mid S \subseteq \Omega \;\&\; S \neq \emptyset\}$ is a set of events.

- Given a set $S \in \Sigma$ and lotteries f and g, $f \preceq_S g$ is intended to mean that "g is at least as good as f, given that the true state of the world is in S."

  If the agent thinks the actual state is in S, then the agent would choose lottery f over g.

- $f \sim_S g$ and $f \prec_S g$ are defined as usual.

Expected Utility Theorem: Notation

- A **conditional-probability function** $p : \Sigma \to \Delta(\Omega)$ is a function that gives the probability of a state t given that an event $S \in \Sigma$ occurs.

- A **utility function** is any function $u : X \times \Omega \to \mathbb{R}$. It is said to be *state independent* iff for all $s, t \in \Omega, u(x, s) = u(x, t)$.

- The **expected utility value** of a lottery f given an even S, $E_p(u(f)|S)$, is calculated as

$$E_p(u(f)|S) = \sum_{t \in S} p(t|S) \sum_{x \in X} u(x, t) f(x|t)$$

Expected Utility Theorem

Let f, g be lotteries and S and event.

1. *(Completeness)* $f \succeq_S g$ or $g \succeq_S f$.

2. *(Transitivity)* If $f \succeq_S g$ and $g \succeq_S h$ then $f \succeq_S h$.

3. *(Relevance)* If, for all $t \in \Omega$, $f(\cdot|t) = g(\cdot|t)$ then $f \sim_S g$.

4. *(Monotonicity)* If $f \succ_S g$ and $0 \leq \beta < \alpha \leq 1$ then $\alpha f + (1-\alpha)g \succeq_S \beta f + (1-\beta)g$.

5. *(Continuity)* If $f \succeq_S g$ and $g \succeq_S h$ then there exists a number γ such that $0 \leq \gamma \leq 1$ and $g \sim_S \gamma f + (1-\gamma)h$.

Expected Utility Theorem

6. *(Objective substitution)* If $e \succeq_S f$, $h \succeq_S g$ and $0 \leq \alpha \leq 1$, then $\alpha e + (1-\alpha)h \succeq_S \alpha f + (1-\alpha)g$.

7. *(Subjective substitution)* If $g \succeq_S f$, $g \succeq_T f$ and $S \cap T = \emptyset$ then $g \succeq_{S \cup T} f$.

8. *(Non-triviality)* For all $t \in \Omega$ there exist $x, y \in X$ such that $[x] \succ_{\{t\}} [y]$

9. *(State Neutrality)* For all $r, t \in \Omega$, if $f(\cdot|r) = f(\cdot|t)$, $g(\cdot|r) = g(\cdot|t)$ and $g \succeq_{\{r\}} f$, then $g \succeq_{\{t\}} f$.

Expected Utility Theorem

6. *(Objective substitution)* If $e \succ_S f$, $h \succeq_S g$ and $0 < \alpha \leq 1$, then $\alpha e + (1-\alpha)h \succ_S \alpha f + (1-\alpha)g$.

7. *(Subjective substitution)* If $g \succ_S f$, $g \succ_T f$ and $S \cap T = \emptyset$ then $g \succ_{S \cup T} f$.

8. *(Non-triviality)* For all $t \in \Omega$ there exist $x, y \in X$ such that $[x] \succeq_{\{t\}} [y]$

9. *(State Neutrality)* For all $r, t \in \Omega$, if $f(\cdot|r) = f(\cdot|t)$, $g(\cdot|r) = g(\cdot|t)$ and $g \succeq_{\{r\}} f$, then $g \succeq_{\{t\}} f$.

Expected Utility Theorem

6. *(Objective substitution)* If $e \succeq_S f$, $h \succeq_S g$ and $0 \leq \alpha \leq 1$, then $\alpha e + (1-\alpha)h \succeq_S \alpha f + (1-\alpha)g$.

7. *(Subjective substitution)* If $g \succeq_S f$, $g \succeq_T f$ and $S \cap T = \emptyset$ then $g \succeq_{S \cup T} f$.

8. *(Non-triviality)* For all $t \in \Omega$ there exist $x, y \in X$ such that $[x] \succ_{\{t\}} [y]$

9. *(State Neutrality)* For all $r, t \in \Omega$, if $f(\cdot|r) = f(\cdot|t)$, $g(\cdot|r) = g(\cdot|t)$ and $g \succeq_{\{r\}} f$, then $g \succeq_{\{t\}} f$.

Expected Utility Theorem

Axioms 1 - 8 are jointly satisfied iff there exists a utlity function $u : X \times \Omega \to \mathbb{R}$ and a conditional probability function $p : \Sigma \to \Delta(\Omega)$ such that

1. For all $t \in \Omega$, $\max_{x \in X} u(x,t) = 1$ and $\min_{x \in X} u(x,t) = 0$.
2. For all R, S, T such that $R \subseteq S \subseteq T \subseteq \Omega$ and $S \neq \emptyset$, $p(R|T) = p(R|S)p(S|T)$.
3. For all $f, g \in L$ and $S \in \Sigma$, $f \succeq_S g$ iff $E_p(u(f)|S) \succeq E_p(u(g)|S)$.

If, furthermore, axiom 8 is also satisfied, then u is state-independent.

Limitations of the Expected Utility Model

Allais Problem: Let $X = \{\$12 \text{ million}, \$1 \text{ million}, \$0\}$, and let

$$\begin{aligned} f_1 &= .10\ [\$12 \text{ million}] + .90\ [\$0] \\ f_2 &= .11\ [\$1 \text{ million}] + .89\ [\$0] \\ f_3 &= [\$1 \text{ million}] \\ f_4 &= .10\ [\$12 \text{ million}] + .89\ [\$1 \text{ million}] + .01\ [\$0] \end{aligned}$$

Which would you choose?

Limitations of the Expected Utility Model

Allais Problem: Let $X = \{\$12 \text{ million}, \$1 \text{ million}, \$0\}$, and let

$$\begin{aligned} f_1 &= .10\ [\$12 \text{ million}] + .90\ [\$0] \\ f_2 &= .11\ [\$1 \text{ million}] + .89\ [\$0] \\ f_3 &= [\$1 \text{ million}] \\ f_4 &= .10\ [\$12 \text{ million}] + .89\ [\$1 \text{ million}] + .01\ [\$0] \end{aligned}$$

Which would you choose?

Allais, M., 1953, Le comportement de l'homme rationnel devant le risque: critique des postulats et axiomes de l'école Américaine, *Econometrica* **21**, 503-546.

Limitations of the Expected Utility Model

Ellsberg Paradox: Suppose an urn contains 30 red balls and 60 other balls that are either black or yellow.

Limitations of the Expected Utility Model

Ellsberg Paradox: Suppose an urn contains 30 red balls and 60 other balls that are either black or yellow.

Gamble A: You receive \$100 if you draw a red ball
Gamble B: You receive \$100 if you draw a black ball

or

Limitations of the Expected Utility Model

Ellsberg Paradox: Suppose an urn contains 30 red balls and 60 other balls that are either black or yellow.

Gamble A: You receive \$100 if you draw a red ball
Gamble B: You receive \$100 if you draw a black ball

or

Gamble C: You receive \$100 if you draw a red or yellow ball
Gamble D: You receive \$100 if you draw a black or yellow ball

Which would you choose?

Limitations of the Expected Utility Model

Ellsberg Paradox: Suppose an urn contains 30 red balls and 60 other balls that are either black or yellow.

Gamble A: You receive \$100 if you draw a red ball
Gamble B: You receive \$100 if you draw a black ball

or

Gamble C: You receive \$100 if you draw a red or yellow ball
Gamble D: You receive \$100 if you draw a black or yellow ball

Which would you choose?

Ellsberg, D. (1961) "Risk, Ambiguity, and the Savage Axioms" *Quarterly Journal of Economics* **75**: 643-669.

Limitations of the Expected Utility Model

Newcomb's Paradox:

Suppose there is a **Predictor**, an entity somehow presented as being exceptionally skilled at predicting people's actions. The Predictor is completely infallible and incapable of error.

Limitations of the Expected Utility Model

Newcomb's Paradox:

Suppose there is a **Predictor**, an entity somehow presented as being exceptionally skilled at predicting people's actions. The Predictor is completely infallible and incapable of error.

There are two opaque boxes, labeled A and B. The player is faced

with the choice

1. Take the contents of both boxes, or
2. Take the contents of only box B.

Limitations of the Expected Utility Model

Newcomb's Paradox:

Box A contains \$1,000.

Limitations of the Expected Utility Model

Newcomb's Paradox:

Box A contains \$1,000.

The contents of box B: At some point before the start of the game, the Predictor makes a prediction as to whether the player of the game will take just box B, or both boxes.

Limitations of the Expected Utility Model

Newcomb's Paradox:

Box A contains \$1,000.

The contents of box B: At some point before the start of the game, the Predictor makes a prediction as to whether the player of the game will take just box B, or both boxes.

If the Predictor predicts that both boxes will be taken, then box B will contain nothing.

Limitations of the Expected Utility Model

Newcomb's Paradox:

Box A contains \$1,000.

The contents of box B: At some point before the start of the game, the Predictor makes a prediction as to whether the player of the game will take just box B, or both boxes.

If the Predictor predicts that both boxes will be taken, then box B will contain nothing.

If the Predictor predicts that only box B will be taken, then box B will contain \$1,000,000.

Limitations of the Expected Utility Model

"To almost everyone, it is perfectly clear and obvious what should be done. The difficulty is that these people seem to divide almost evenly on the problem, with large numbers thinking that the opposing half is just being silly."

Nozick, Robert (1969), "Newcomb's Problem and Two principles of Choice," in Essays in Honor of Carl G. Hempel, p 115.

Levi, Isaac (1982), "A Note on Newcombmania," Journal of Philosophy 79 (1982): 337-42.

Limitations of the Expected Utility Model

Kahneman and Tversky:

Limitations of the Expected Utility Model

Kahneman and Tversky:

1. You arrive at the theatre, for which you have bought a pair of tickets that cost $40. You suddenly realize that your tickets have fallen out of your pocket and are lost

Limitations of the Expected Utility Model

Kahneman and Tversky:

1. You arrive at the theatre, for which you have bought a pair of tickets that cost $40. You suddenly realize that your tickets have fallen out of your pocket and are lost

2. You are arriving at the theatre for which tickets cost $40. You did not buy the tickets in advance , but you put $40 in your pocket before you left home. You suddenly realize that the $40 has fallen out of your pocket and is lost.

Limitations of the Expected Utility Model

Kahneman and Tversky:

1. You arrive at the theatre, for which you have bought a pair of tickets that cost $40. You suddenly realize that your tickets have fallen out of your pocket and are lost

2. You are arriving at the theatre for which tickets cost $40. You did not buy the tickets in advance , but you put $40 in your pocket before you left home. You suddenly realize that the $40 has fallen out of your pocket and is lost.

What would you do in each situation?

Kahneman, Daniel, and Amos Tversky (1979) "Prospect Theory: An Analysis of Decision under Risk", Econometrica, XVLII (1979), 263-291

End of Part 1

To be continued...