Lecture 3: Some Game Theory

Eric Pacuit

ILLC, University of Amsterdam

staff.science.uva.nl/∼epacuit

epacuit@science.uva.nl

Lecture Date: February 23, 2006

Caput Logic, Language and Information: Social Software

staff.science.uva.nl/∼epacuit/caputLLI.html

Today's Lecture

- Finish the decision theory from last week
- "Just enough" Game Theory
  - Strategic Games
  - Strategic Games: Nash Equilibrium
  - Strictly Competitive Games
  - Extensive Games (with perfect information)
  - Kuhn's Theorem

Game Theory

Game theory is a bag of analytical tools designed to help us understand the phenomena that we observe when decision-makers interact.

A course in game theory by M. Osborne and A. Rubinstein

Game Theory

Game theory is a bag of analytical tools designed to help us understand the phenomena that we observe when decision-makers interact.

A course in game theory by M. Osborne and A. Rubinstein

A **game** is a description of strategic interaction that includes

- actions the players *can* take
- description of the players' interests (i.e., preferences),

Game Theory

Game theory is a bag of analytical tools designed to help us understand the phenomena that we observe when decision-makers interact.

A course in game theory by M. Osborne and A. Rubinstein

A **game** is a description of strategic interaction that includes

- actions the players *can* take
- description of the players' interests (i.e., preferences),

It does not specify the actions that the players do take.

Game Theory

Game theory is a bag of analytical tools designed to help us understand the phenomena that we observe when decision-makers interact.

A course in game theory by M. Osborne and A. Rubinstein

A **game** is a description of strategic interaction that includes

- actions the players *can* take
- description of the players' interests (i.e., preferences),

It does not specify the actions that the players do take.

A *solution* is a systematic description of the outcomes that may emerge in a family of games.

Game Theory: Uncertainty

The players may be

- uncertain about the objective parameters of the environment
- imperfectly informed about events that happen in the game
- uncertain about action of the other players that are not deterministic
- uncertain about the reasoning of the other players

More on this later in the semester....

For this lecture, assume the agents are **perfectly informed**.

Strategic Games

A **strategic games** is a tuple $\langle N, \{A_i\}_{i \in N}, \{\succeq_i\}_{i \in N} \rangle$ where

Let $A = \times_{i \in N} A_i$

- N is a finite set of **players**
- for each $i \in N$, A_i is a nonempty set of **actions**
- for each $i \in N$, $\succeq_i$ is a **preference relation** on A

Strategic Games: Comments on Preferences

- Preferences may be over a set of consequences C. Assume $g : A \to C$ and $\{\succeq_i^* \mid i \in N\}$ a set of preferences on C. Then for $a, b \in A$,

$$a \succeq_i b \text{ iff } g(a) \succeq_i^* g(b)$$

- Consequences may be affected by exogenous random variable whose realization is not known before choosing actions. Let Ω be a set of states, then define $g : A \times \Omega \to C$. Where $g(a|\cdot)$ is interpreted as a lottery.

- Often $\succeq_i$ are represented by **utility functions** $u_i : A \to \mathbb{R}$

Strategic Games: Nash Equilibrium

Let $\langle N, \{A_i\}_{i\in N}, \{\succeq_i\}_{i\in N}\rangle$ be a strategic game

For $a_{-i} \in A_{-i}$, let

$$B_i(a_{-i}) = \{a_i \in A_i \mid (a_{-i}, a_i) \succeq_i (a_{-i}, a_i') \;\forall\; a_i' \in A_i\}$$

B_i is the **best-response** function.

Strategic Games: Nash Equilibrium

Let $\langle N, \{A_i\}_{i \in N}, \{\succeq_i\}_{i \in N} \rangle$ be a strategic game

For $a_{-i} \in A_{-i}$, let

$$B_i(a_{-i}) = \{a_i \in A_i \mid (a_{-i}, a_i) \succeq_i (a_{-i}, a_i') \; \forall \; a_i' \in A_i\}$$

B_i is the **best-response** function.

$a^* \in A$ is a **Nash equilibrium** iff $a_i^* \in B_i(a_{-i}^*)$ for all $i \in N$.

Strategic Games Example: Bach or Stravinsky?

	b_c	s_c
b_r	2,1	0,0
s_r	0,0	1,2

Strategic Games Example: Bach or Stravinsky?

	b_c	s_c
b_r	2,1	0,0
s_r	0,0	1,2

$$N = \{r, c\} \quad A_r = \{b_r, s_r\}, A_c = \{b_c, s_c\}$$

$$A = A_r \times A_c = \{(b, b), (b, s), (s, b), (s, s)\}$$

Strategic Games Example: Bach or Stravinsky?

	b_c	s_c
b_r	2,1	0,0
s_r	0,0	1,2

$$N = \{r, c\} \quad A_r = \{b_r, s_r\}, A_c = \{b_c, s_c\}$$

$$A = A_r \times A_c = \{(b,b), (b,s), (s,b), (s,s)\}$$

$$B_r(b_c) = \{b_r\} \qquad B_r(s_c) = \{s_r\}$$

Strategic Games Example: Bach or Stravinsky?

	b_c	s_c
b_r	2,1	0,0
s_r	0,0	1,2

$$N = \{r, c\} \quad A_r = \{b_r, s_r\}, A_c = \{b_c, s_c\}$$
$$A = A_r \times A_c = \{(b, b), (b, s), (s, b), (s, s)\}$$

$B_r(b_c) = \{b_r\}$ $\qquad$ $B_r(s_c) = \{s_r\}$

$B_c(b_r) = \{b_c\}$ $\qquad$ $B_c(s_r) = \{s_c\}$

Strategic Games Example: Bach or Stravinsky?

	b_c	s_c
b_r	2,1	0,0
s_r	0,0	1,2

$$N = \{r, c\} \quad A_r = \{b_r, s_r\}, A_c = \{b_c, s_c\}$$
$$A = A_r \times A_c = \{(b,b),(b,s),(s,b),(s,s)\}$$

$B_r(b_c) = \{b_r\}$ $\qquad$ $B_r(s_c) = \{s_r\}$

$B_c(b_r) = \{b_c\}$ $\qquad$ $B_c(s_r) = \{s_c\}$

(b_r, b_c) is a Nash Equilibrium $\qquad$ (s_r, s_c) is a Nash Equilibrium

Strategic Games Example: Bach or Stravinsky?

	b_c	s_c
b_r	2,1	0,0
s_r	0,0	1,2

$$N = \{r, c\} \quad A_r = \{b_r, s_r\}, A_c = \{b_c, s_c\}$$
$$A = A_r \times A_c = \{(b, b), (b, s), (s, b), (s, s)\}$$

$B_r(b_c) = \{b_r\}$ $\quad$ $B_r(s_c) = \{s_r\}$

$B_c(b_r) = \{b_c\}$ $\quad$ $B_c(s_r) = \{s_c\}$

(b_r, b_c) is a Nash Equilibrium $\quad$ (s_r, s_c) is a Nash Equilibrium

Strategic Games Example: Mozart or Mahler?

	Mo	Ma
Mo	2,2	0,0
Ma	0,0	1,1

Strategic Games Example: Prisoner's Dilema

	D	C
D	3,3	0,4
C	4,0	1,1

Strategic Games Example: Hawk-Dove

	D	H
D	3,3	1,4
H	4,1	0,0

Strategic Games Example: Matching Pennies

	H	T
H	1,-1	-1,1
T	-1,1	1,-1

Existence of Nash Equilibrium

Theorem The strategic game $\langle N, \{A_i\}_{i \in N}, \{\succeq_i\}_{i \in N} \rangle$ has a Nash equilibrium if for all $i \in N$

- the set A_i is a nonempty compact convex subset of a Euclidian space
- the preference relations $\succeq_i$ are continuious and quasi-concave on A_i

Kakutani's Fixed Point Theorem

Let X be a compact convex subset of $\mathbb{R}^n$ and let $f : 2^X \to 2^X$ be a function such that

- for all $x \in X$, $f(x)$ is nonempty and convex
- the graph of f is closed (for all sequences (x_n) and (y_n) such that $y_n \in f(x_n)$ for all n, $\lim_n x_n = x$ and $\lim_n y_n = y$, we have $y \in f(x)$.

Then there exists $x^* \in X$ such that $x^* \in f(x^*)$

Strictly Competitive Games

A strategic game $\langle \{1, 2\}, \{A_i\}, \{\succeq_i\}\rangle$ if for any $a \in A$ and $b \in A$ we have $a \succeq_1 b$ iff $b \succeq_2 a$.

If preferences are represented by utility functions u_1 and u_2. Then the game is called **zerosum** and we assume $u_1 + u_2 = 0$.

Strictly Competitive Games

A strategic game $\langle\{1,2\},\{A_i\},\{\succeq_i\}\rangle$ if for any $a \in A$ and $b \in A$ we have $a \succeq_1 b$ iff $b \succeq_2 a$.

If preferences are represented by utility functions u_1 and u_2. Then the game is called **zerosum** and we assume $u_1 + u_2 = 0$.

	H	T
H	1,-1	-1,1
T	-1,1	1,-1

Maxminimizer

Assumption: *player i will choose an action that is best for him on the assumption that whatever he does, player j will hurt him as much as possible.*

i **maxminimizes** his choice.

Maxminimizer

Assumption: *player i will choose an action that is best for him on the assumption that whatever he does, player j will hurt him as much as possible.*

i **maxminimizes** his choice.

Claim In a strictly competitive game, a pair of strategies is a Nash equilibrium iff the action of each player is a maxminizer.

Furthermore, in games that possess Nash equilibria all equilibria yield the **same** payoff.

Maxminizier: Formal Definition

Let $\langle \{1,2\}, \{A_i\}, \{\succeq_i\} \rangle$ be a strictly competitive strategic game. The action $x^* \in A_1$ ($y^* \in A_2$) is a **maxminizer for player 1 (player 2)** if

$$\min_{y \in A_2} u_1(x^*, y) \geq \min_{y \in A_2} u_1(x, y) \text{ for all } \; x \in A_1$$

$$\min_{x \in A_1} u_2(x, y^*) \geq \min_{x \in A_1} u_2(x, y) \text{ for all } \; y \in A_2$$

Maxminizier: Formal Definition

Let $\langle \{1,2\}, \{A_i\}, \{\succeq_i\} \rangle$ be a strictly competitive strategic game. The action $x^* \in A_1$ ($y^* \in A_2$) is a **maxminizer for player 1 (player 2)** if

$$\min_{y \in A_2} u_1(x^*, y) \geq \min_{y \in A_2} u_1(x, y) \text{ for all } \; x \in A_1$$

$$\min_{x \in A_1} u_2(x, y^*) \geq \min_{x \in A_1} u_2(x, y) \text{ for all } \; y \in A_2$$

Lemma Let $\langle \{1,2\}, \{A_i\}, \{\succeq_i\} \rangle$ be a strictly competitive strategic game. Then

$$\max_{y \in A_2} \min_{x \in A_1} u_2(x, y) = - \min_{y \in A_2} \max_{x \in A_1} u_1(x, y)$$

Further, $y \in A_2$ solves $\max_{y \in A_2} \min_{x \in A_1} u_2(x, y)$ iff it solves $\min_{y \in A_2} \max_{x \in A_1} u_1(x, y)$

Strictly Competitive Games: Nash Equilibrium

Lemma Let $\langle \{1,2\}, \{A_i\}, \{u_i\} \rangle$ be a strictly competitive strategic game. Assume (x^*, y^*) is a Nash equilibrium. Then

Strictly Competitive Games: Nash Equilibrium

Lemma Let $\langle \{1, 2\}, \{A_i\}, \{u_i\} \rangle$ be a strictly competitive strategic game. Assume (x^*, y^*) is a Nash equilibrium. Then

1. x^* is a maxminimizer for player 1 and y^* is a maxminimizer for player 2.

Strictly Competitive Games: Nash Equilibrium

Lemma Let $\langle \{1,2\}, \{A_i\}, \{u_i\} \rangle$ be a strictly competitive strategic game. Assume (x^*, y^*) is a Nash equilibrium. Then

1. x^* is a maxminimizer for player 1 and y^* is a maxminimizer for player 2.

2. $\max_x \min_y u_1(x,y) = \min_y \max_x u_1(x,y) = u_1(x^*, y^*)$. *Thus all N.E. yield the same payoff.*

Strictly Competitive Games: Nash Equilibrium

Lemma Let $\langle \{1,2\}, \{A_i\}, \{u_i\} \rangle$ be a strictly competitive strategic game. Assume (x^*, y^*) is a Nash equilibrium. Then

1. x^* is a maxminimizer for player 1 and y^* is a maxminimizer for player 2.

2. $\max_x \min_y u_1(x,y) = \min_y \max_x u_1(x,y) = u_1(x^*, y^*)$. *Thus all N.E. yield the same payoff.*

Further if $\max_x \min_y u_1(x,y) = \min_y \max_x u_1(x,y)$ and x^* is a maxminimizer for 1 and y^* is a maxminimizer for 2, then (x^*, y^*) is a Nash equilibrium.

Extensive Games with Perfect Information

In strategic games, strategies are chosen *once and for all at the start of the game*

Extensive Games with Perfect Information

In strategic games, strategies are chosen *once and for all at the start of the game*

Extensive games are explicit descriptions of the **sequential structure** of the decision problem encountered by the players in a strategic situation.

Extensive Games with Perfect Information

- A set N of players

- A set H is a set of **sequences**, or **histories**, that is "closed" and contains all finite prefixes. I.e.,

  - The empty sequence is in H

  - If $(a_k)_{k=1,\ldots,K} \in H$ and $L < K$ then $(a_k)_{k=1,\ldots L} \in H$

  - If an infinite sequence $(a_k)_{k=1}^{\infty}$ satisfies $(a_k)_{k=1,\ldots,L} \in H$ for each $L \geq 1$, then $(a_k)_{k=1}^{\infty} \in H$

Extensive Games with Perfect Information

- Let $Z \subseteq H$ be the set of **terminal** histories.

- A function $P : H - Z \rightarrow N$

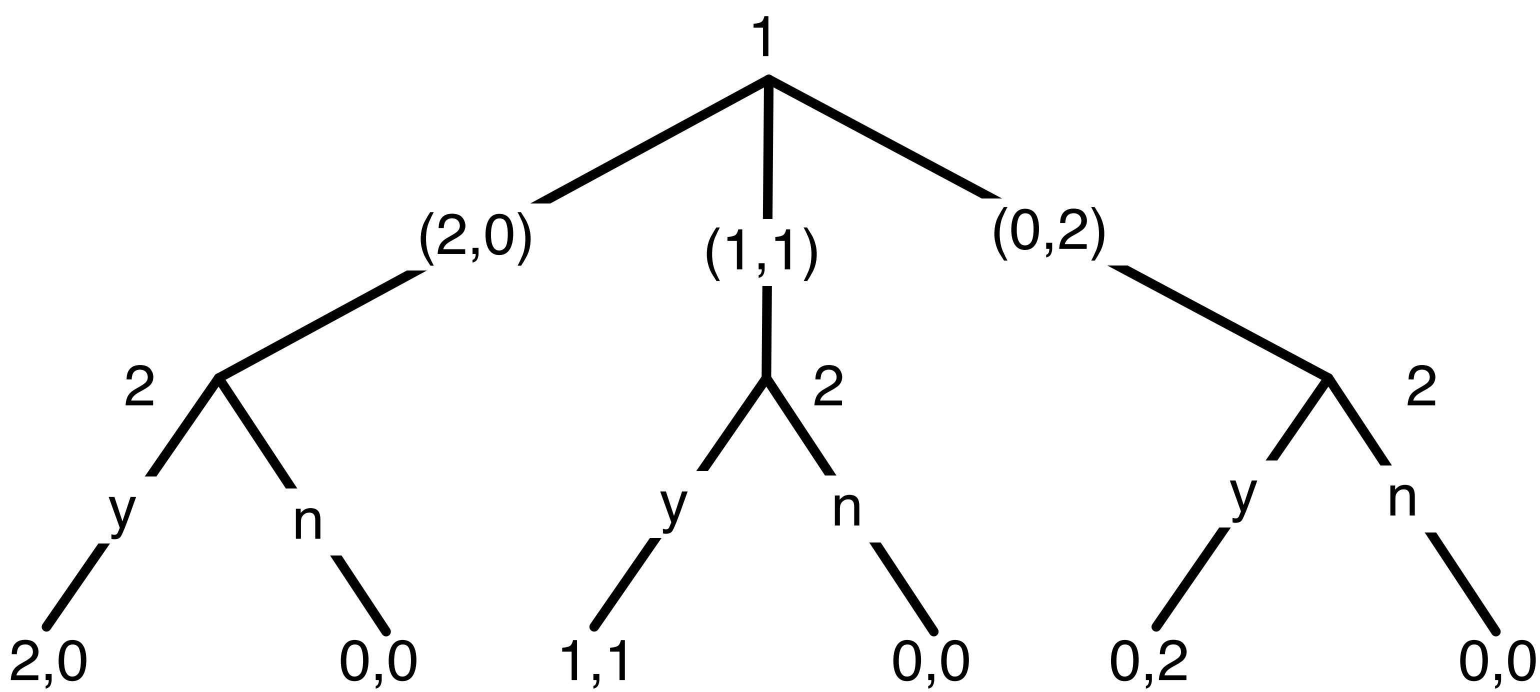
- For each $i \in N$, a relation $\succeq_i$ on Z.

- For an nonterminal history h, let $A(h) = \{a \mid (h, a) \in H\}$

Extensive Games with Perfect Information: Example

Suppose two people use the following procedure to share two desirable goods. One of them proposes an allocation, which the other accepts or rejects. In the event of rejection, neither person receives either of the objects. Each person cares only about the number of objects he obtains.

Extensive Games with Perfect Information: Example



Extensive Games with Perfect Information: Strategies

A **strategy of player** $i \in N$ is a *partial* function $\sigma_i : H - Z \to A$ where for each $h \in H - Z$, $\sigma(h) \in A(h)$ and $P(h) = i$.

A strategy specifies an action chosen by a player for **every** *history after which it is his turn to move, even histories that may never be reached.*

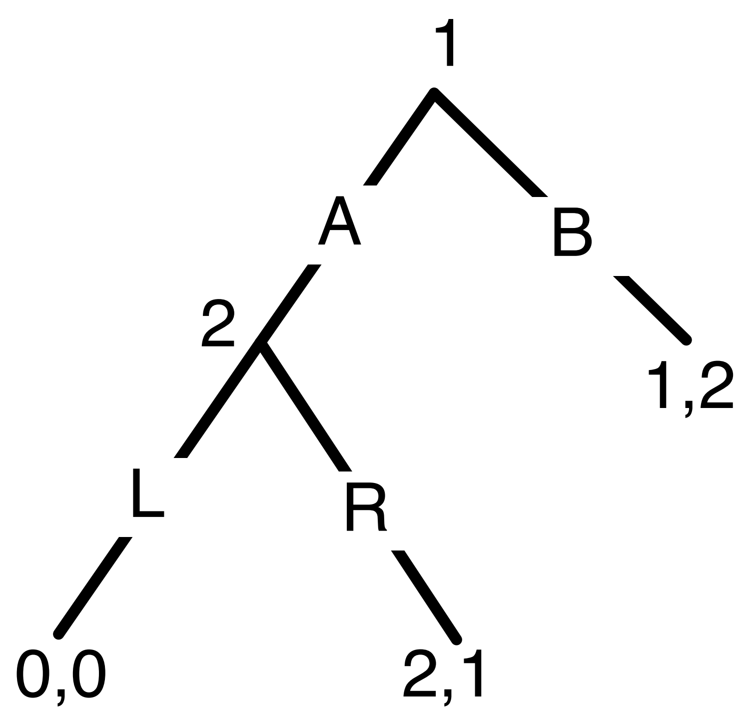
Given a strategy profile $\sigma = (\sigma_i)_{i \in N}$, the **outcome** $O(\sigma)$ is the terminal history that results when each player i follows σ_i.

Extensive Games with Perfect Information: Nash Equilibrium

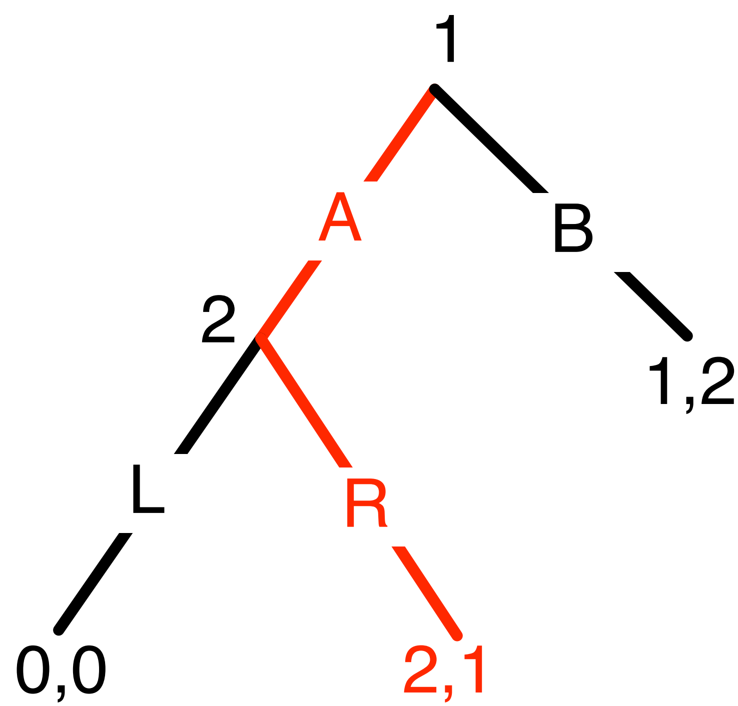
A **Nash Equilibrium** of an extensive game with perfect information is a strategy profile σ^* such that for every player $i \in N$ we have

$$O(\sigma^*_{-i}, \sigma^*_i) \succeq_i O(\sigma^*_{-i}, \sigma_i) \text{ for every strategy } \sigma_i \text{ of player } i$$

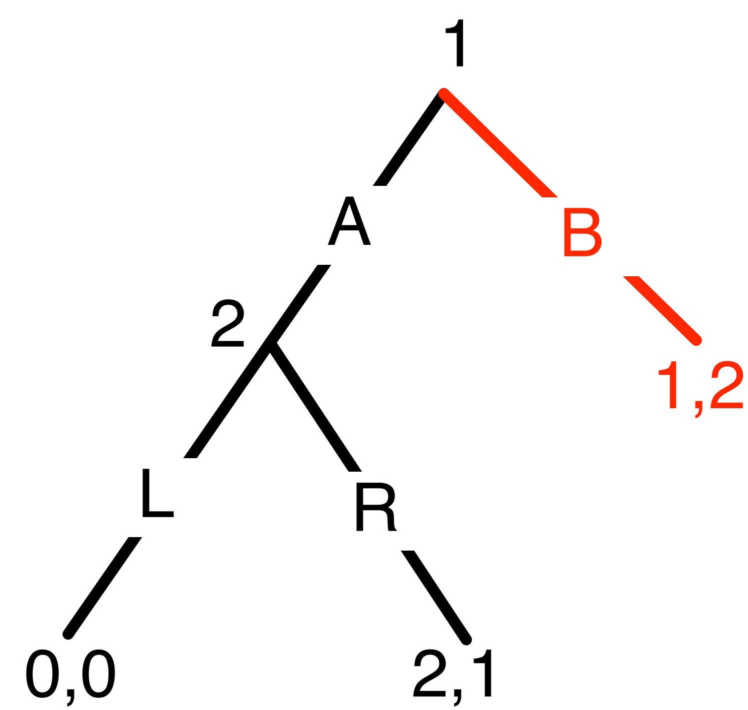
Extensive Games with Perfect Information: Example



Extensive Games with Perfect Information: Nash Equilibrium



Extensive Games with Perfect Information: Nash Equilibrium



Subgame Perfect Equilibrium

The **subgame of the extensive game with perfect information** $G = \langle N, H, P, (\succeq_i)\rangle$ that follows history h is the game

$$G_h = \langle N, H_h, P_n, (\succeq_i^h)\rangle$$

where

- H_h is the set of sequences of h' of actions for which $(h, h') \in H$
- P_h is defined $P|_h(h') = P(h, h')$ for each $h' \in H_h$
- $\succeq_i^h$ is define $h' \succeq_i^h h''$ iff $(h, h') \succeq_i (h, h'')$
- Given a strategy σ, let σ^h denote the history induced on G_h by σ
- Let O_h denote the outcome function of G_h.

Subgame Perfect Equilibrium

A strategy profile σ is a subgame perfect equilibrium of a game G if for every $i \in N$ and every nonterminal history h for which $P(h) = i$ we have

$$O_h(\sigma_{-i}^h, \sigma_i^h) \succeq_i^h O_h(\sigma_{-i}^h, \tau_i)$$

for every strategy τ_i of player i in the subgame G_h.

That is, σ^* is a subgame perfect equilibrium if σ^h is a Nash Equilibrium in every subgame of G.

Subgame Perfect Equilibrium

Kuhn's Theorem Every finite extensive game with perfect informaiton has a subgame perfect equilibrium.

Subgame Perfect Equilibrium

An extensive game is a **finite horizon** game if the length of each history is finite.

Lemma (The one deviation property) Let G be a finite horizon extensive game. The strategy profile σ is a subgame perfect equilibrium of G iff for every player $i \in N$ and every history $h \in H$ for which $P(h) = i$ we have

$$O_h(\sigma^h_{-i}, \sigma^h_i) \succeq^h_i O_h(\sigma^h_{-i}, \tau_i)$$

for every strategy τ_i of player i in the subgame G_h that differs from σ^h_i only in the action it prescribes after the initial history of G_h.

We have left out A LOT....

- Mixed Strategies, Bayesian Games
- Mechanism Design
- Rationalizibility and Iterated Elimination of Dominated Actions
- Correlated Equilibrium
- Imperfect Information Games
- Repeated Games
- Bargaining Games
- Auctions
- Cooperative Game Theory

We have left out A LOT....

- Mixed Strategies, Bayesian Games (will be discussed)
- Mechanism Design (will be discussed)
- Rationalizibility and Iterated Elimination of Dominated Actions (may be discussed)
- Correlated Equilibrium (may not be discussed)
- Imperfect Information Games (will be discussed)
- Repeated Games (will not be discussed)
- Bargaining Games (see Ulle's course)
- Auctions (see Ulle's course)
- Cooperative Game Theory (may be discussed, K. Apt's course)

We have left out A LOT....

- Mixed Strategies, Bayesian Games (will be discussed)
- Mechanism Design (will be discussed)
- Rationalizibility and Iterated Elimination of Dominated Actions (may be discussed)
- Correlated Equilibrium (may not be discussed)
- Imperfect Information Games (will be discussed)
- Repeated Games (will not be discussed)
- Bargaining Games (see Ulle's course)
- Auctions (see Ulle's course)
- Cooperative Game Theory (may be discussed, K. Apt's course)