Lecture 5: Fair Division Algorithms

Eric Pacuit

ILLC, University of Amsterdam

staff.science.uva.nl/~epacuit

epacuit@science.uva.nl

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Caput Logic, Language and Information: Social Software

staff.science.uva.nl/~epacuit/caputLLI.html

Plan for Today

Discuss some fair division algorithms

- What does it mean to “fairly” divide goods?
- Indivisible Goods
- Divisible Goods (Cutting a Cake)
  - Divide and Choose
  - Banach-Knaster Last Diminisher
  - Dubins-Spanier Moving Knife Procedure

Fairness Conditions

- **Proportional:** (for two players) each player receives at least 50% of their valuation.

- **Envy-Free:** no party is willing to give up its allocation in exchange for the other player's allocation, so no players envies anyone else.

- **Equitable:** each player values its allocation the same *according to its own valuation function.*

- **Efficiency:** there is no other division better for everybody, or better for some players and not worse for the others

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Truthfulness

Some procedures ask players to represent their preferences.

This representation need not be "truthful"

Typically, it is assumed that agents will follow a maximin strategy (maximize the set of items that are guaranteed)

Indivisible Goods

Steven J. Brams, Paul H. Edelman and Peter C. Fishburn, "Paradoxes of Fair Division", *Journal of Philosophy*, **98:6**, (June): 300-314.

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- Players cannot compensate each other with side payments
- All players have positive values for every item
- A player prefers a set S to different set T if
  - S has as many elements as T
  - for every item in $t \in T - S$ there is a distinct item $s \in S - T$ that the player prefers to t.

Indivisible Goods: Envy-Freeness and Efficiency

A unique envy-free division may be inefficient

$$\begin{array}{lcccccc} A: & 1 & 2 & 3 & 4 & 5 & 6 \\ B: & 4 & 3 & 2 & 1 & 5 & 6 \\ C: & 5 & 1 & 2 & 6 & 3 & 4 \end{array}$$

$$A: \{1, 3\}$$

$$B: \{2, 4\}$$

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This is the unique *envy-free* outcome.

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The division $(12, 34, 56)$ pareto-dominates the above division

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However, $(12, 34, 56)$ is **not** (necessarily) envy-free

Indivisible Goods: Envy-Freeness and Efficiency

A unique envy-free division may be inefficient

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$$A : \{1, 3\}$$

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There is no other division, including an efficient one, that guarantees envy-freeness.

Indivisible Goods: Envy-Freeness and Efficiency

There may be no envy-free division, even when all players have different preference rankings

Indivisible Goods: Envy-Freeness and Efficiency

There may be no envy-free division, even when all players have different preference rankings

Trivial if all players have the same preference.

Indivisible Goods: Envy-Freeness and Efficiency

There may be no envy-free division, even when all players have different preference rankings

$$\begin{array}{llll} A: & 1 & 2 & 3 \\ B: & 1 & 3 & 2 \\ C: & 2 & 1 & 3 \end{array}$$

Three divisions are efficient: $(1,3,2)$, $(2,1,3)$ and $(3,1,2)$.
However, none of them are envy-free.

Indivisible Goods: Envy-Freeness and Efficiency

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Three divisions are efficient: $(1, 3, 2)$, $(2, 1, 3)$ and $(3, 1, 2)$.
However, none of them are envy-free.

In fact, there is **no** envy-free division.

Divisible Goods

When good can be divided, then there is more hope to find a division that satisfies the above properties.

There is much literature focused on algorithms to fairly divide a cake between $2, 3, \ldots$ people.

Cutting a Cake: Divide and Choose

Procedure: one player cuts the cake into two portions and the other player chooses one.

Suppose A is the cutter.

If A has no information about the other player's preferences, then A should cut the cake at some point x so that the value of the portion to the left of x is equal to the value of the portion to the right.

This strategy creates an envy-free allocation, but it is not necessarily equitable.

Cutting a Cake: Divide and Choose

Assume two players A and B value a cake along a line from $x = 0$ to $x = 1$.

Each player has a value function $v_A(x)$ and $v_B(x)$ where $v_A(x) \geq 0$ and $v_B(x) \geq 0$ for $x \in [0, 1]$

The total valuations of the players — the areas under $v_A(x)$ and $v_B(x)$ — are 1

Only parallel, vertical cuts, perpendicular to the horizontal x-axis are made

Cutting a Cake: Divide and Choose

Suppose A values the vanilla half twice as much as the chocolate half. Hence,

$$v_A(x) = \begin{cases} 4/3 & x \in [0, 1/2] \\ 2/3 & x \in (1/2, 1] \end{cases}$$

$$v_B(x) = \begin{cases} 1/2 & x \in [0, 1/2] \\ 1/2 & x \in (1/2, 1] \end{cases}$$

A should cut the cake at $x = 3/8$:

$$(4/3)(x - 0) = 4/3(1/2 - x) + 2/3(1 - 1/2)$$

Note that the portions are not equitable (B receive $5/8$ according to his valuation)

Cutting a Cake: Banach-Knaster Last Diminisher Procedure

Suppose there are n different agents: $p_1, \ldots, p_n$.

Procedure:

- The first person (p_1) cuts out a piece which he claims is his fair share.

- Then, the piece goes around being inspected, in turn, by persons $p_2, p_3, \ldots, p_n$.

  - Anyone who thinks the piece is not too large just passes it. Anyone who thinks it is too big, may reduce it, putting some back into the main part.

Cutting a Cake: Banach-Knaster Last Diminisher Procedure

- After the piece has been inspected by p_n, the last person who reduced the piece, takes it. If there is no such person, i.e., no one challenged p_1 , then the piece is taken by p_1.

- The algorithm continues with $n - 1$ participants.

This procedure is equitable but not envy-free

Cutting a Cake: Dubins-Spanier Moving-Knifer Procedure

Procedure: A referee holds a knife at the left edge of the cake and slowly moves it across the cake so that it remains parallel to its starting position.

At any time, any one of the three players (A, B or C) can call “cut”.

When this occurs, the player who called cut receives the piece to the left of the knife and exits the game.

Cutting a Cake: Dubins-Spanier Moving-Knifer Procedure

The game now continues moving until a second player calls cut.

The second player receives the second piece and the third player gets the remainder.

If either two or three players call cut at the same time, the cut piece is given to one of the callers at random.

This procedure is equitable but not envy-free

How about some pie, and other results

See *Fair Division: From cake-cutting to dispute resolution* by S. Brams and A. Taylor for more information.

Brams and Fishburn "Fair Division of Indivisible Items between Two People with Identical Preferences: Envy-freeness, Pareto-optimality, and Equity." *Social Choice and Welfare* **17:2**, (February) pgs. 247 — 267.

Brams, Jones and Klamler, "Better Ways to Cut a Cake",

Barbanel and Brams, "Cutting a Pie Is Not a Piece of Cake"

Information about the paper will be available shortly on the website

Next week: Guest Lecture by K. Apt

Next^2 week: From Reasoning about Programs to Reasoning about Games

Next^3 week: No class (exam week)