Lecture 7: Introduction to Interactive Epistemology

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Caput Logic, Language and Information: Social Software

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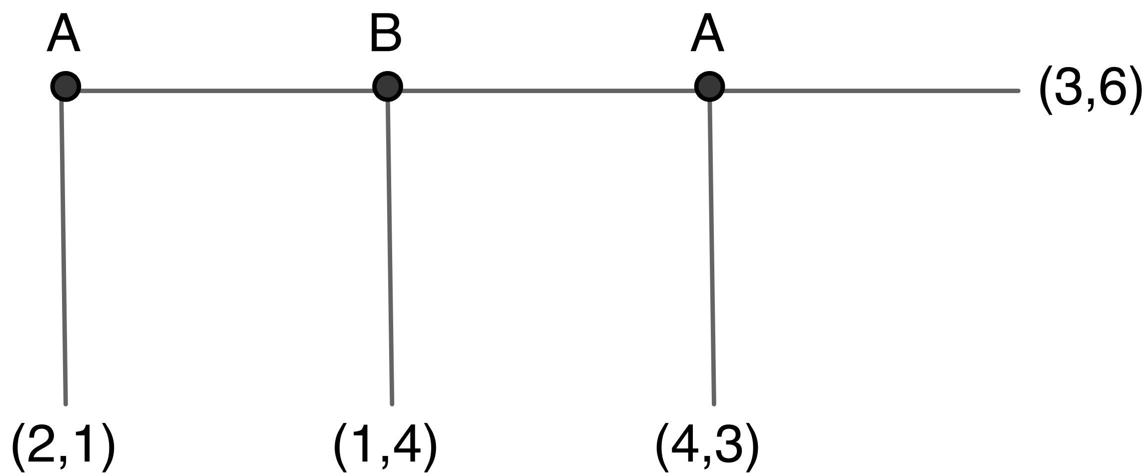
Overview

- Introduction and Motivating Examples

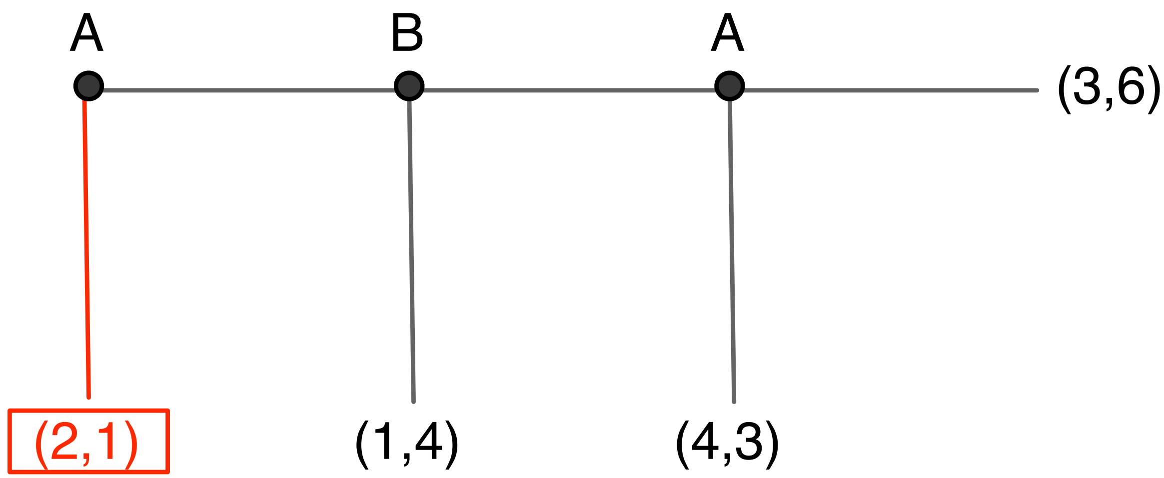
- Formalizing Beliefs ([Bonanno and Battigalli, 1999, Aumann 1999, Brandenburger 2005])

- Universal Type Spaces
  - Brief Overview of the Literature
  - An Impossibility Result: [Brandenburger and Keisler, 2004]

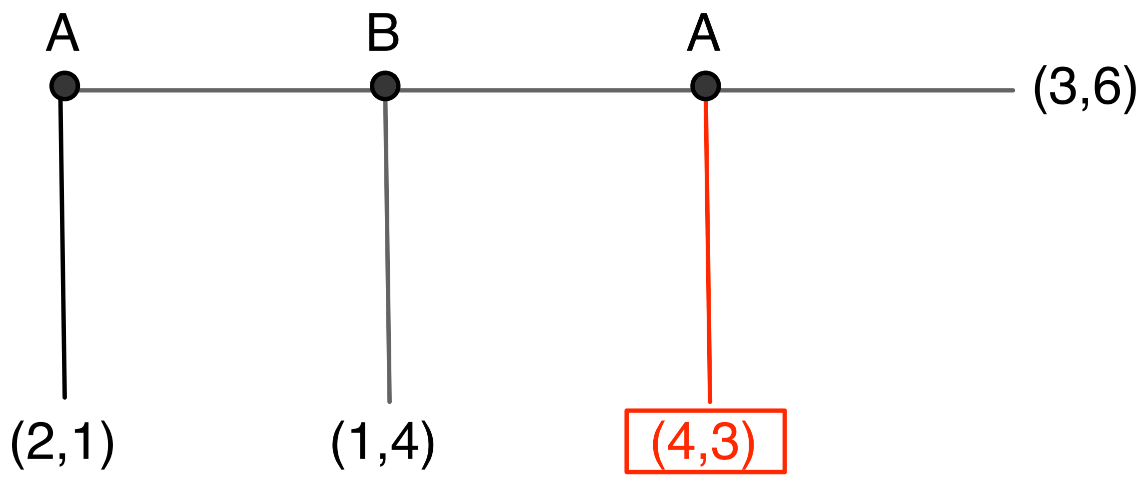
Puzzle 1



The Backwards Induction Solution



But what if...



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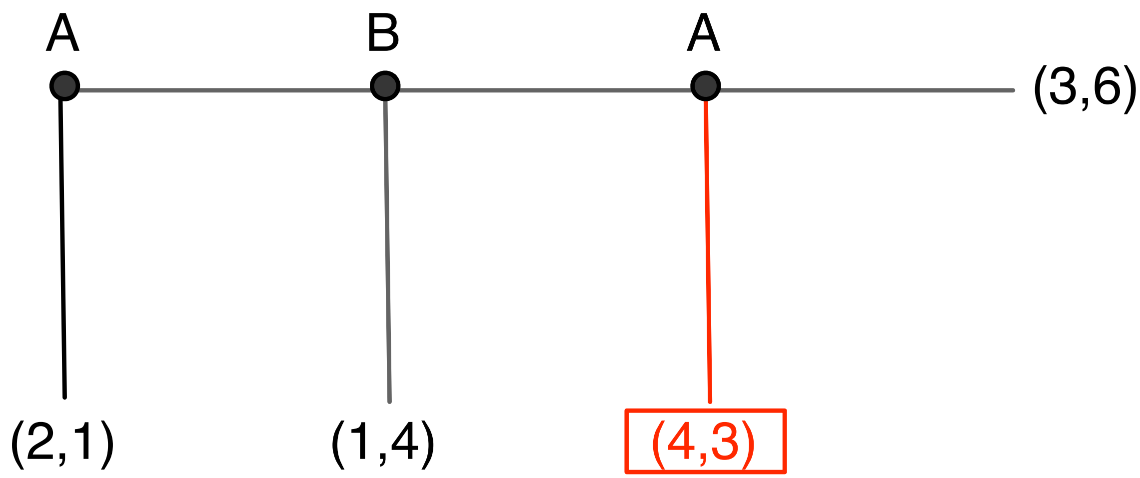

- Are the players *irrational*?
- What argument leads to the BI solution?

Puzzle 2

	L	R
T	$1, 1$	$0, 0$
M	$1, 1$	$2, 1$
B	$0, 0$	$2, 1$

Iterated Admissibility (IA)

	L	R
T	$1,1$	$0,0$
M	$1,1$	$2,1$
B	$0,0$	$2,1$

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Iterated Admissibility (IA)

	L	R
T	1, 1	0, 0
M	1, 1	2, 1

Iterated Admissibility (IA)

	L
T	$1, 1$
M	$1, 1$

Iterated Admissibility (IA)

	L
T	$1, 1$
M	$1, 1$

Rationale: Suppose Ann is rational in the sense that she will not play any inadmissible (weakly dominated) strategies. If she thinks Bob is rational in the same way, then she can remove from consideration any of Bob's strategies that are inadmissible. So, Ann will choose only a strategy that is admissible in the submatrix that results from removing (some) of Bob's inadmissible strategies. And so on reaching the IA set.

Iterated Admissibility (IA)

	L
T	$1, 1$
M	$1, 1$

A Problem: The argument for deletion of a weakly dominated strategy for player i is that he contemplates the possibility that every strategy combination of his rivals occurs with positive probability. However, this hypothesis clashes with the logic of iterated deletion, which assumes, precisely, that eliminated strategies are not expected to occur. [Mas-Colell, Whinston and Green, 1995, pg. 282]

Fundamental Problem

What does it mean to say that the players in a game are **rational**, each **thinks** each other is rational, each thinks each other thinks the others are rational, etc.

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What does it mean to say that the players in a game are **rational**, each **thinks** each other is rational, each thinks each other thinks the others are rational, etc.

Epistemic Program in Game Theory: An explicit description of the players' beliefs is part of the descripiton of a game.

Identify for any game the strategies that are chosen by rational and intelligent players who know the strucutre of the game, the preference of the other players and recognize each others rationality and knowledge.

Describing Beliefs

- **Infinite Regress?** i's choice depends on j's choice, which depends on i's *beliefs* about what j will choose, which depends on j's *beliefs* about i's *beliefs* about ...

- **Harsanyi Type Spaces:** The main tool used to describe beliefs. Originated in Harsanyi's seminal work on games of *incomplete* information [Harsanyi 1967-1968].

- **Knowledge of Possibility Structures:** What is assumed that the agents know about the structure used to represent the uncertainty of the agents?

Impossibility Result

Belief models and languages are artifacts created by the analyst to describe a strategic situation. *Shouldn't these artifacts be thought of as, in some sense, available to the players themselves?*

- ([Brandenburger and Keisler, 2004]) Basic limitation to the analysis of games: *If the analyst's tools are available to the players, there are statements that the players can think about but cannot assume. The model must be incomplete.*

Describing Beliefs

Fix a set of possible states (complete descriptions of a situation) Ω.
Two main approaches to describe beleifs:

- Set-theortical (Kripke Structures, Aumann Structures): For each state and each agent i, specify a set of states that i considers possible.

- Probabilistic (Bayesian Models, Harsanyi Type Spaces): For each state, define a (subjective) probability function over the set of states for each agent.

Bayesian Frame

$$\mathcal{B} = \langle \mathcal{A}, \Omega, \{p_i\}_{i \in \mathcal{A}} \rangle$$

- $\mathcal{A}$ is a finite set of players
- Ω is a nonempty set of states
- for each $i \in \mathcal{A}$, $p_i : \Omega \to \Delta(\Omega)$ such that for all $\alpha, \beta \in \Omega$

$$\text{if } p_{i,\alpha}(\beta) > 0 \text{ then } p_{i,\beta} = p_{i,\alpha}$$

$\Delta(\Omega)$ is the set of probability measures on Ω. If Ω is infinite then the standard measure-theoretic assumptions apply.

We write $p_{i,\alpha}$ instead of $p_i(\alpha)$.

Belief Frames

$$\mathcal{F} = \langle \mathcal{A}, \Omega, \{R_i\}_{i \in \mathcal{A}} \rangle$$

- $\mathcal{A}$ is a finite set of players
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- for each $i \in \mathcal{A}$, $R_i \subseteq \Omega \times \Omega$ (we may think of R_i as a function $R_i : \Omega \to 2^W$ where $R_i(\omega) = \{v \mid wR_iv\}$)

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Given a Bayesian frame $\mathcal{B} = \langle \mathcal{A}, \Omega, \{p_i\}_{i \in \mathcal{A}} \rangle$, we can generate a belief frame

$$\overline{\mathcal{B}} = \langle \mathcal{A}, \Omega, \{P_i\}_{i \in \mathcal{A}} \rangle$$

where $P_i : \Omega \rightarrow 2^\Omega$ is defined as

$$P_i(\omega) = supp(p_{i,\omega})$$

Models

Let S be a set of external states whose interpretation is given. A (Bayesian, Belief) model is a tuple $\mathcal{M}_B = \langle \mathcal{F}, \sigma \rangle$ where $\sigma : \Omega \to S$.

An event $E \subseteq \Omega$ is **true** at state w if $w \in E$.

We define $\mathsf{B}_i : 2^\Omega \to 2^\Omega$ as follows $\mathsf{B}_i(E) = \{w \mid B_i(w) \subseteq E\}$.

$\mathsf{B}_i(E)$ is the event that i believes event E.

Hierarchies of Knowledge (Belief Frames)

Fix a state $w \in \Omega$.

- **Ground Facts** $\Phi_0 = \{E \mid w \in E$ and E a ground event$\}$
- i**'s first order beliefs:** $\Phi_i^1 = \{\mathsf{B}_i(E) \mid E$ a ground event$\}$, $\Phi^1 = \Phi^0 \cup \cup_{i \in \mathcal{A}} \Phi_i^1$
- i**'s second order beliefs:** $\Phi_i^2 = \{\mathsf{B}_i(E) \mid E \in \Phi^1\}$, $\Phi^2 = \Phi^1 \cup \cup_{i \in \mathcal{A}} \Phi_i^2$
- i**'s** n**-th order beliefs:** $\Phi_i^n = \{\mathsf{B}_i(E) \mid E \in \Phi^{n-1}\}$, $\Phi^n = \phi^{n-1} \cup \cup_{i \in \mathcal{A}} \phi_i^{n-1}$.

Thus each state in a belief models corresponds to the following infinite hierarchy of beliefs

$$(\Phi_0, (\Phi_i^1, \Phi_i^2, \ldots)_{i \in \mathcal{A}})$$

Can this be done with Bayesian frames?

First Order Beliefs: $\mu^1_{i,\omega} \in \Delta(S)$ such that for all $s \in S$

$$\mu^1_{i,\omega}(s) = \sum_{\omega' \in \sigma^{-1}(s)} p_{i,\omega}(\omega')$$

Second Order Belief: $\mu^2_{i,\omega} \in \Delta(S \times [\Delta(S)]^{n-1})$ such that for all $s \in S$

$$\mu^2_{i,\omega}(s, (\mu'_j)_{i \neq j}) = \sum_{\omega' : (\sigma(\omega'), (\mu'_j)_{i \neq j}) = (s, (\mu'_j)_{i \neq j})} p_{i,\omega}(\omega')$$

Given any state ω we have the following infinite hierarchy of beliefs

$$(\sigma(\omega), (\mu^1_{i,\omega}, \mu^2_{i,\omega}, \ldots)_{i \in \mathcal{A}})$$

Two Consistency Conditions

Given any Bayesian model and state $w \in \Omega$, the hierarchy $(\sigma(\omega), (\mu^1_{i,\omega}, \mu^2_{i,\omega}, \ldots)_{i \in \mathcal{A}})$ satisfies the following properties

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Coherency $mrg_S \mu^2_{i,t} = \mu^1_{i,t}$ and similarly for higher-order beliefs. That is for all $k > 1$, $\mu^{k-1}_{i,t}$ is the marginal of $\mu^k_{i,t}$ on the appropriate space.

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Common Knowledge of Coherency Higher order probabilities assign probability zero to lower order hierarchies of other individuals violating Coherency.

Harsanyi Type Space

$$\mathcal{T} = \langle \mathcal{A}, S, \{T_i\}_{\in \mathcal{A}}, \{\lambda_i\}_{i \in \mathcal{A}} \rangle$$

- $\mathcal{A}$ is a finite set of n agents
- S is a set of states
- T_i is a set of types
- $\lambda_i : T_i \to \Delta(S \times T_{-i})$

A state of the world is a tuple

$$(s, t_1, \ldots, t_n) \in S \times T_1 \times \cdots \times T_n$$

Harsanyi Type Spaces

For simplicity, we assume $S = \times_{i \in \mathcal{A}} S_i$, where each S_i is a strategy space for agent i in some fixed game G. In this case, $\lambda_i : T_i \rightarrow \Delta(S_{-i} \times T_{-i})$.

A fixed state $(s_1, t_1, s_2, t_2, \ldots, s_n, t_n)$ specifies the strategies and each player's *entire hierarchy of beliefs*:

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1. i's first-order beliefs: $T_i \mapsto \Delta(S_{-i} \times T_{-i}) \mapsto \Delta(S_{-i})$ (marginalizing)

2. i's second-order beliefs: $T_i \mapsto \Delta(S_{-i} \times T_{-i}) \mapsto \Delta(S^{-i} \times \times_{i \neq j} \Delta(S_{-j} \times T_{-j})) \mapsto \Delta(S_{-i} \times \times_{j \neq i} \Delta(S_{-j}))$ (marginalizing)

An Example

	L	R
U	2, 2	0, 0
D	0, 0	1, 1

$\lambda_r(t_r)$

u_c	0	1/2
t_c	0	1/2
	L	R

$\lambda_r(u_r)$

u_c	1/2	0
t_c	0	1/2
	L	R

$\lambda_c(t_c)$

u_r	0	1/2
t_r	0	1/2
	U	D

$\lambda_c(u_c)$

u_r	1/2	0
t_r	0	1/2
	U	D

State: (D, t_r, R, t_c)

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$\lambda_c(u_c)$

u_r	1/2	0
t_r	0	1/2
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r is **correct** about c's strategy (similarly, for c).

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$\lambda_c(u_c)$

u_r	$1/2$	0
t_r	0	$1/2$
	U	D

r thinks it is possible c is wrong about her strategy

An Example

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u_c	0	1/2
t_c	0	1/2
	L	R

$\lambda_r(u_r)$

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$\lambda_c(t_c)$

u_r	0	1/2
t_r	0	1/2
	U	D

$\lambda_c(u_c)$

u_r	1/2	0
t_r	0	1/2
	U	D

r is **rational**. (Similarly for c)

An Example

	L	R
U	$2,2$	$0,0$
D	$0,0$	$1,1$

$\lambda_r(t_r)$

u_c	0	1/2
t_c	0	1/2
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$\lambda_c(t_c)$

u_r	0	1/2
t_r	0	1/2
	U	D

$\lambda_c(u_c)$

u_r	1/2	0
t_r	0	1/2
	U	D

r thinks it is possible that c is **irrational**.

Iteratively Undominated Strategies

Iteratively delete all **strongly** dominated strategies.

- A **rational player** will not play a strongly dominated strategy.

- If player i assigns probability 1 to the rationality of the other players, then i's marginal on the other players' strategies will assign probability 1 to undominated strategies.

- So, a player that is rational and believes the other players are rational will not play a strategy that becomes dominated after the first round of deletions, and so on.

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Type structures are needed to make this argument formal.

Formalizing Rationality

- (s_i, t_i) is **rational** if s_i maximizes i's expected payoff under the marginal on S_{-i} of the measure $\lambda_i(t_i)$
- t_i **believes** an event $E \subseteq S_{-i} \times T_{-i}$ if $\lambda_i(t_i)(E) = 1$, and write

$$\mathsf{B_i}(E) = \{t_i \in T_i \mid t_i \text{ believes } E\}$$

- For each player i, let R_i^1 be the set of all rational pairs (s_i, t_i), and for $m > 0$ define R_i^m inductively by

$$R_i^{m+1} = R_i^m \cap [S_i \times \mathsf{B}_i(R_{-i}^m)]$$

- $(s_1, t_1, \ldots, s_n, t_n) \in R^{m+1}$ there is **rationality and** m**th-order belief of rationality** $(RmBR)$ at this state. If $(s_1, t_1, \ldots, s_n, t_n) \in \cap_{m=1}^{\infty} R^m$ there is **rationality and common belief of rationality** $(RCBR)$ at this state.

Formal Result

Fix a type structure and a state $(s_1, t_1, \ldots, s_n, t_n)$ *at which there is* $RCBR$*. Then the strategy profile* $(s_1, \ldots, s_n)$ *is* IU*. Conversely, fix an* IU *profile* $(s_1, \ldots, s_n)$*. There is a type structure and a state* $(s_1, t_1, \ldots, s_n, t_n)$ *at which there is* $RCBR$*.*

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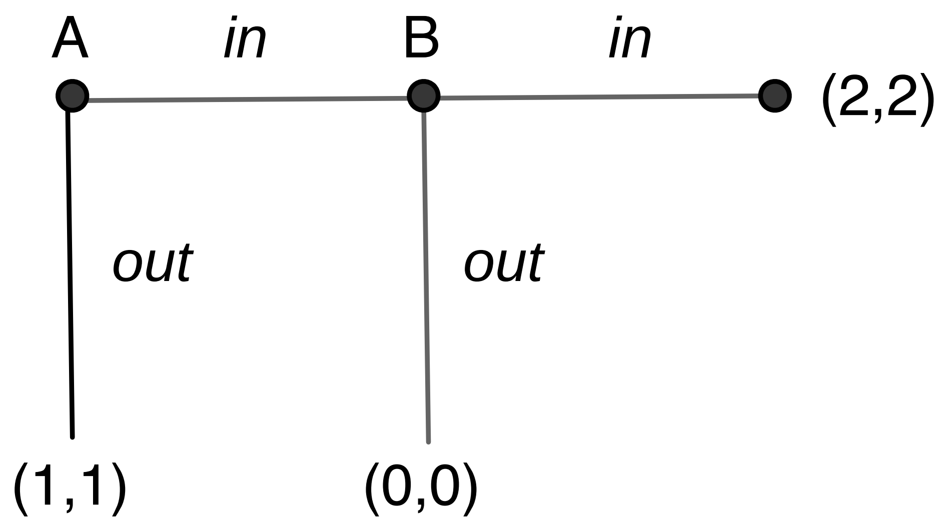
The epistemic program in game theory works by refining these notions of rationality and common belief in rationality ($RCBR$) to yield various solution concepts.

A Problem: Probability Zero Events



$\lambda_A(t_A)$

t_B	1	0
	out	*in*

$\lambda_B(t_B)$

t_A	1	0
	out	*in*

A Question

- For any given set S of external states we can use a Bayesian model or a type space on S to provide consistent representations of the individuals' sysetms of beliefs.

- Every state in a model or type space induces an infinite hierarchy of beliefs, but *not all consistent infinite hierarchies are in any finite model.* It is not obvious that even in an infinite model that all consistent hierarchies of beliefs can be represented.

- Which type space is the “correct” one to work with?

Is there a universal type space?

A **universal type space** is a types space to which every type space (on the same space of states of nature and same set of agents) can be mapped, preferably always in a unique way, by a map that preserves the structure of the type space.

If such a space exists, then the any analysis of a game could be carried out in this space without the risk of missing any "relevant" states of the world.

Yes, if ...

The existence of a universal types space depends on the topological and/or measure theoretic assumptions being made about the underlying state space S.

First shown by Mertens and Zamir (1985)

The problem is to define the set of all infinite hierarchies of beliefs satsifying the same consistency properties (coherency and common knowledge of coherency) as that of hierarchies obtained at some state in a type space.

Kolomogorov Extension Theorem

Without topological or measure-theoretic assumptions, there is an impossibility theorem ([Brandenburger and Keisler, 2004])

Why do we care?

Given a game-theoretic situation, what does it means for a player to be rational and to "recognize" the rationality of the other players?

Ideally, answer the above question should imply that agents will follow strategies suggested by well-known solution concepts (i.e., Nash equilibrium, backward induction, iterated removal of strictly (weakly) dominated strategies).

It turns out that finding the connection between rationality, what agents think about the situation and what actually happens depends on the existence of a "rich enough" space of types, i.e., a universal type space. ([Battigalli and Siniscalchi, 1999], [Brandenburger and Keisler, 2000], [Brandenbureger, Freidenburg and Keisler, 2004])

Canonical, Complete and Terminal Models

1. **Canonical models**: Start with a space of underlying uncertainty, players form beliefs over this space, believes over this space and the space of 0-th order beliefs, and so on inductively. The question is, does this process end?

2. **Complete models**: The "two-way subjectivity" models described later.

3. **Terminal models**: Given a category $\mathbf{C}$ of models of beliefs, call a model $\mathcal{M}$ in $\mathbf{C}$ terminal if for any other model $\mathcal{N}$ in $\mathbf{C}$, there is a unique belief preserving morphism from $\mathcal{M}$ to $\mathcal{N}$.

Proposal: ([Brandneburger and Keisler, 2004] *A general framework to study the relationship between canonical, complete and terminal models (absent specific structure).*

Overview of the Literature

- Introduces Type Spaces: [Harsanyi, 1967]

- Existence proofs (under various topological assumptions): [Armbruster and Boge, 1979], [Mertens and Zamir, 1985], [Brandenburger and Dekel, 1993], [Heifetz, 1993], [Heifetz and Samet, 1998], [Battigalli and Siniscalchi, 1999], [Meier, 2002], [Salonen, 2003]

- Impossibility Result: [Brandenburger and Kesiler, 2004], [Meier, 2005]

- Knowledge structures: [Fagin, Halpern and Vardi, 1991], [Heifetz and Samet, 1998], [Fagin, Geanokoplos, Halpern and Vardi , 1999]

- Relevant surveys: [Aumann, 1999], [Bonanno and Battigalli, 1999], [Brandenburger, 2002], [Brandenburger, 2005]

- Logic of type spaces: [Heifetz and Mongin, 2001], [Meier, 2001]

A Puzzle

Ann believes that Bob assumes* that
Ann believes that Bob's assumption is wrong.

Does Ann believe that Bob's assumption is wrong?

* An **assumption** (or strongest belief) is a belief that implies all other beliefs.

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Main Result

No belief model can be complete for a language that contains first-order logic

Belief Model: a set of states for each player, and a relation for each player that specifies when a state of one player considers a state of the other player to be possible.

Language: the language used by the players to formulate their beliefs

Complete: A belief model is complete for a language if every statement in a player's langauge which is possible (i.e. true for some states) can be assumed by the player.

Open Question

Can we find a logic such that

1. Complete belief models for exist for each game;

2. notions such as rationality, belief in rationality, etc. are expressible in ; and

3. the ingredients in 1 and 2 can be combined to yield various well-known game-theoretic solution concepts.

Belief Models

A **belief model** is a structure

$$\mathcal{M} = (U^a, U^b, P^a, P^b, \ldots)$$

where

- U^a, U^b are nonempty sets
- $P^a \subsetneq U^a \times U^b$ and $P^b \subsetneq U^b \times U^a$
- the following sentences are true (the relations are serial)

$$\forall x \exists y P^a(x, y) \qquad \forall y \exists x P^b(y, x)$$

**x and y range over U^a and U^b respectively.

Beliefs and Assumptions

x **believes** a set $Y \subseteq U^b$ if $\{y \mid P^a(x,y)\} \subseteq Y$

x **assumes** a set $Y \subseteq U^b$ if $\{y \mid P^a(x,y)\} = Y$

Languages

A language for a player should be a set of statements that the player can think about.

Given a belief model $\mathcal{M}$. A **language for** $\mathcal{M}$ is a set $\mathcal{L} = \mathcal{L}^a \cup \mathcal{L}^b$ where $\mathcal{L}^a \subseteq 2^{U^a}$ and $\mathcal{L}^b \subseteq 2^{U^b}$

$\mathcal{M}$ is **complete for** a language $\mathcal{L}$ if each nonempty $Y \in \mathcal{L}^b$ is assumed by some $x \in U^a$, and each nonempty $X \in \mathcal{L}^a$ is assumed by some $y \in U^b$.

Impossibility Results, [Brandenburger and Keislef, 2004]

- **Proposition** No belief model is complete for its power-set language

- **Theorem** Let $\mathcal{M}$ be a belief model and let contain the first-order language for $\mathcal{M}$. Then $\mathcal{M}$ cannot be complete for .

  - **Lemma** Suppose that in $\mathcal{M}$, $\forall y P^a(x_1, y)$ and

    $$x_2 \;\; believes \;\; [y \;\; believes \;\; [x \;\; believes \; \forall x P^b(y, x)]]$$

    Then x_2 $believes$ $\neg P^b(y, x_2)$

  - **Lemma** Suppose that $\mathcal{M}$ is a belief model. Then no element $x_0 \in U^a$ believes the formula

    $$y \;\; assumes \;\; [x \;\; believes \; \neg P^b(y, x)]$$

Proof of the Theorem

$\mathcal{M}$ has a **hole** at a set $Y \subseteq U^b$ if $Y \neq \emptyset$ but is not *assumed* by any element (of U^a)

$\mathcal{M}$ has a **big hole** at a set $Y \subseteq U^b$ if $Y \neq \emptyset$ but is not *believed* by any element (of U^a)

Every belief model has either

1. A hole at U^a or U^b; or
2. A big hole at $\forall x P^b(y, x)$; or
3. A big hole at $x \;\; believes\; \forall x P^b(y, x)$; or
4. A big hole at $y \;\; believes \;\; [x \;\; believes\; \forall x P^b(y, x)]$; or
5. A hole at $x \;\; believes\; \neg P^b(y, x)$; or
6. A big hole at $y \;\; assumes \;\; [x \;\; believes\; \neg P^b(y, x)]$; or

Two Positive Results

1. The notion of assumption is essential. There are strategic models which are **weakly complete** in the sense that every statement which is possible can be believed.

2. The existence problem for complete models is intrinsically multi-player in nature. There exist **semi-complete** belief models. A model $\mathcal{M}$ is semi-complete for a if every nonempty subset of U^b is assumed by some element of U^a.

Topologically Complete Models

Complete models exist for the fragment of first-order logic that does not contain the negation symbol, i.e., the **positive formulas**

The proof using topological methods proceeds in two steps:

1. **Theorem** Assume that the underlying (strategy) space is compact metrizable, then construct an appropriate model $\mathcal{M}$ that is complete for the **compact language** for $\mathcal{M}$ (the set of all compact subsets of the constructed topological space).
2. **Theorem** Every positive formula defines a compact set in the product $U^a \times U^b$.

Select a topic for your paper!

Next week: No class (exam week)