Lecture 9: Part I: More on Game Logic

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Caput Logic, Language and Information: Social Software

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Quick Review

Syntax:

$$\gamma \;:=\; g \mid \phi? \mid \gamma;\gamma \mid \gamma \cup \gamma \mid \gamma^* \mid \gamma^d$$

$$\phi \;:=\; \bot \mid p \mid \neg\phi \mid \phi \vee \phi \mid \langle\gamma\rangle\phi \mid [\gamma]\phi$$

Effectivity Function: For each $g \in \Gamma_0$, $E_g : W \to 2^{2^W}$

Truth in a model: $\mathcal{M}, w \models \langle\gamma\rangle\phi$ iff $(\phi)^{\mathcal{M}} \in E_\gamma(w)$

- $E_{\gamma_1;\gamma_2}(Y) \;:=\; E_{\gamma_1}(E_{\gamma_2}(Y))$
- $E_{\gamma_1\cup\gamma_2}(Y) \;:=\; E_{\gamma_1}(Y) \cup E_{\gamma_2}(Y)$
- $E_{\phi?}(Y) \;:=\; (\phi)^{\mathcal{M}} \cap Y$
- $E_{\gamma^d}(Y) \;:=\; \overline{E_\gamma(\overline{Y})}$
- $E_{\gamma^*}(Y) \;:=\; \mu X.Y \cup E_\gamma(X)$

Game Logic: Untyped Games

$$G = (H, P_A, W_A, W_D, \delta)$$

- A run q is **terminal** if $q = \langle q_0, \cdots, q_n \rangle \in H$ and either (1) $n = \infty$ or there in no q_{n+1} such that $\langle q_0, \cdots, q_{n+1} \rangle \in H$. Let $H^t \subseteq H$ be the set of terminal runs and $H^\infty \subseteq H^t$ be the set of infinite runs.

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- $P_A \subseteq \overline{(H^t)}$ is the set of nonterminal positions where it is **Angel's turn to move**. At all other nonterminal positions it is Demon's turn to move.

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- $P_A \subseteq \overline{(H^t)}$ is the set of nonterminal positions where it is **Angel's turn to move**. At all other nonterminal positions it is Demon's turn to move.

- $W_A \subseteq H^t$ denotes the set of terminal runs **won by Angel**, while $W_D \subseteq H^t$ denotes the set of terminal runs **won by Demon**.

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- A *strategy* for Angel is a function $\sigma : P_A \rightarrow H$ such that $\sigma(\langle q_0, \cdots, q_n \rangle) = \langle q_0, \cdots, q_n, k \rangle$.

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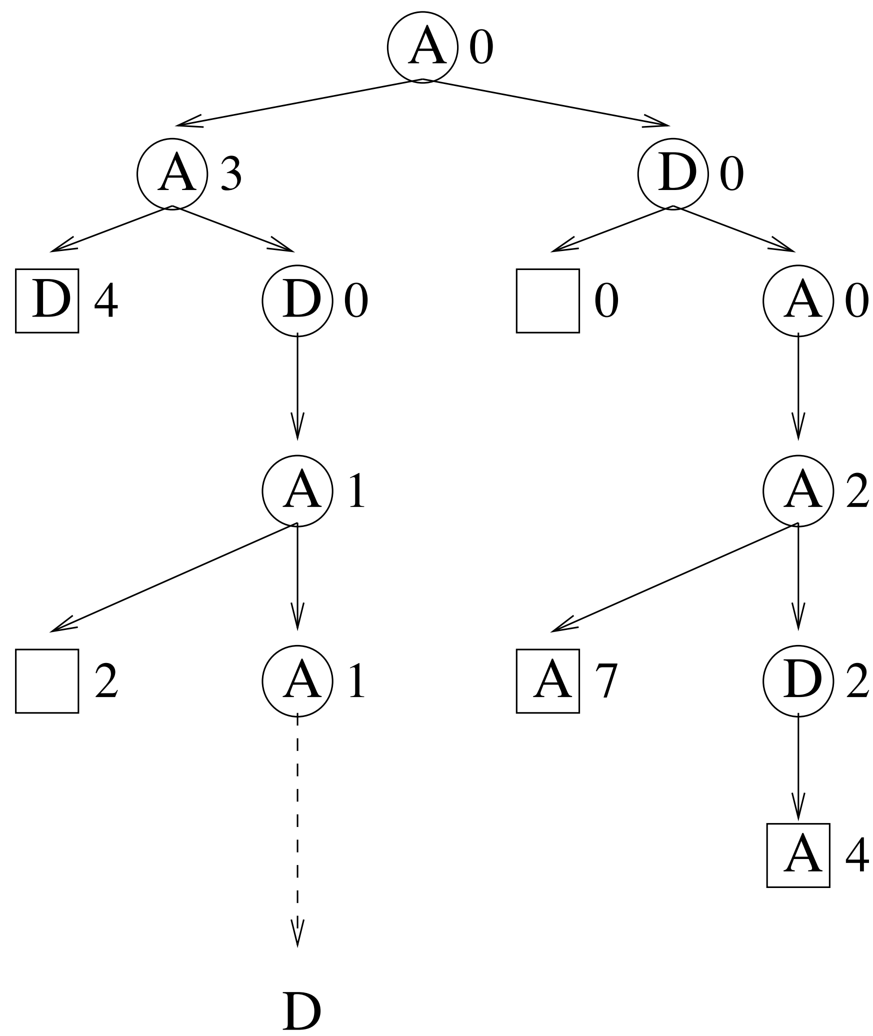
- For any $Q \subseteq H$, let $Q_\sigma = \{q \in Q \mid q \text{ obeys } \sigma\}$.

- For each $X \subseteq S$ and strategy σ for Angel is an X**-strategy** iff for all runs $q \in H^t$ obeying σ, either (1) $q \in W_A$ or (2) $q \notin W_A \cup W_D$ and $\delta(q) \in X$.

Example



Game Logic: Semantics II

Game Web

$$\mathcal{I} = (S, \{G(g,s) | g \in \Gamma_0 \text{ and } s \in S\}, V)$$

$G(g,s)$ are determined untyped games on S such that the initial game position $\langle\rangle$ is associated with state s

Complex games $G(\gamma, s)$ can be defined inductively

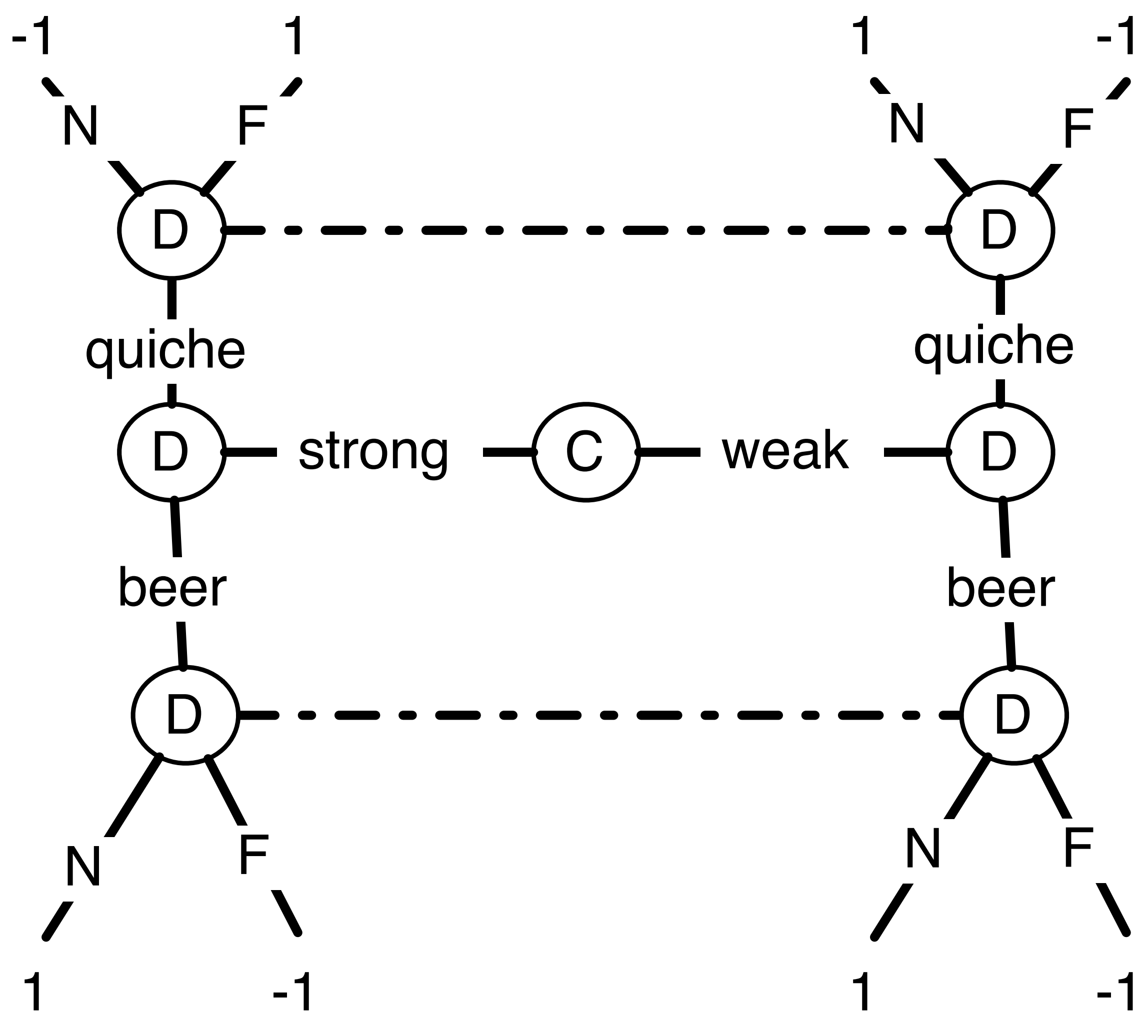
Truth is defined as usual:

1. $\mathcal{I}, s \models \langle\gamma\rangle\phi$ iff Angel has a $(\phi)^{\mathcal{I}}$-strategy in game $G(\gamma, s)$
2. $\mathcal{I}, s \models [\gamma]\phi$ iff Demon has a $(\phi)^{\mathcal{I}}$-strategy in game $G(\gamma, s)$

Example: Beer of Quiche?

Two men, Al and Dick, meet in a bar. Dick is not particularly fond of Al and considers challenging him to a fight. Al can either drink a beer (and get drunk) or eat quiche. Subsequently, Dick has to decide whether or not to fight Al. What complicates the situation is that Dick is unsure of Al's strength. We assume that if Al is weak, then he prefers not to fight, whereas if he is strong, he prefers to fight if and only if he is sober. We want the game to be strictly competitive, so Dick's preferences are the opposite: if Al is weak, Dick prefers to fight him, but if Al is strong, Dick only wants to fight Al if Al is drunk.

M. Osborne and A. Rubinstein. *A Course in Game Theory*. Example 244.2, page 244.


-1
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beer
D
D
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F
1
-1
1
-1

Example: Beer of Quiche?

Let $\Gamma = \{b, q, f, n\}$ be the atomic games:

- b means Al can drink a beer,
- q means Al can eat quiche,
- f means Dick can fight, and
- n means Dick can do nothing (not fight).

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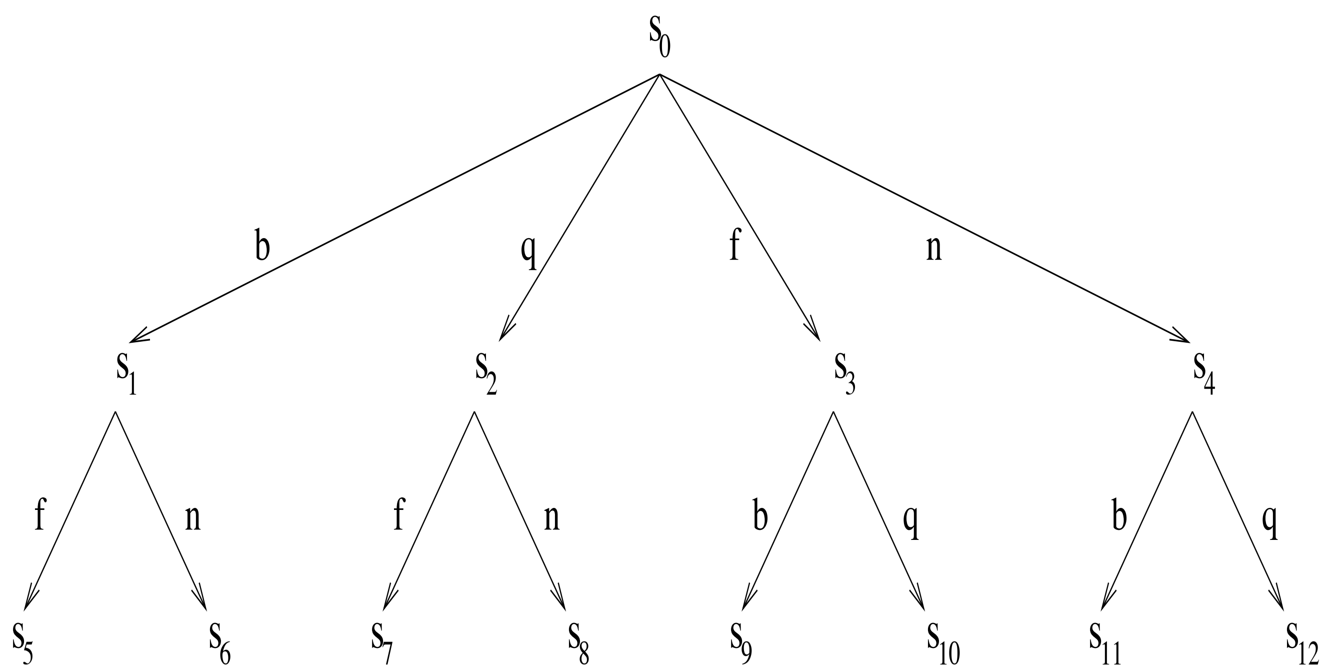

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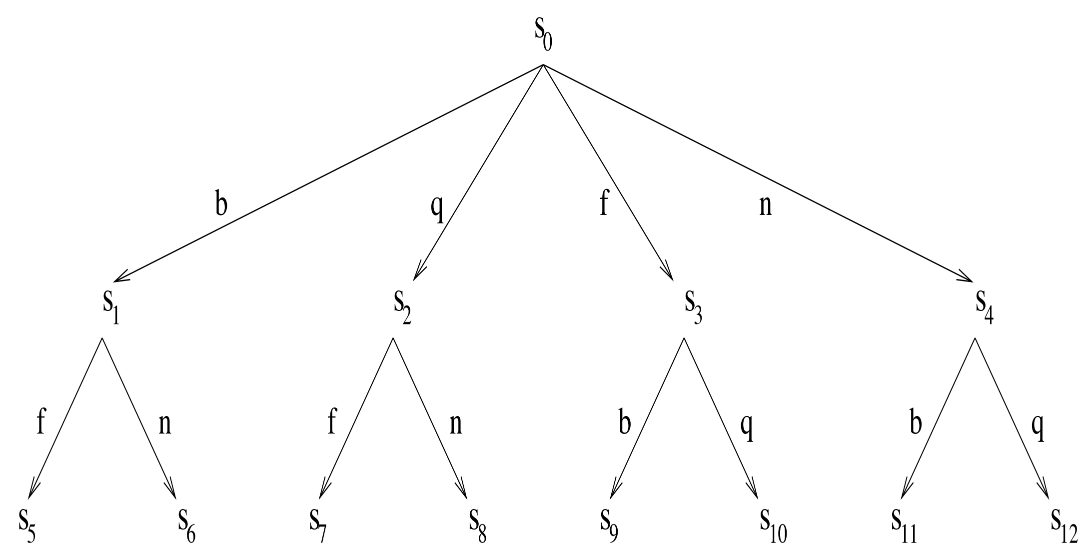
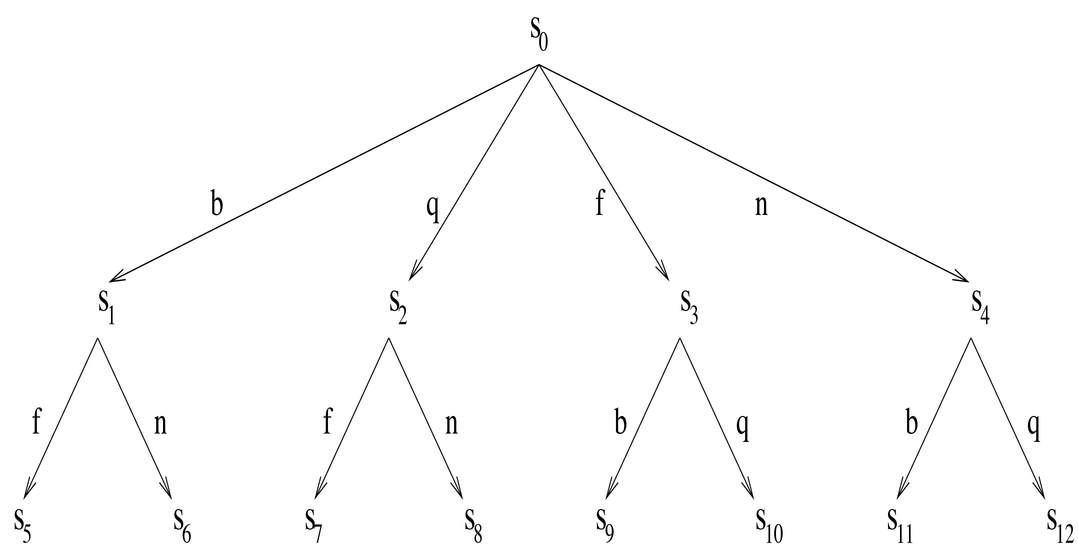


Let S mean that Al is strong. So,

$$\mathsf{S}^{\mathcal{I}} = \{s_6, s_7, s_9, s_{10}\}$$

Example: Beer of Quiche?

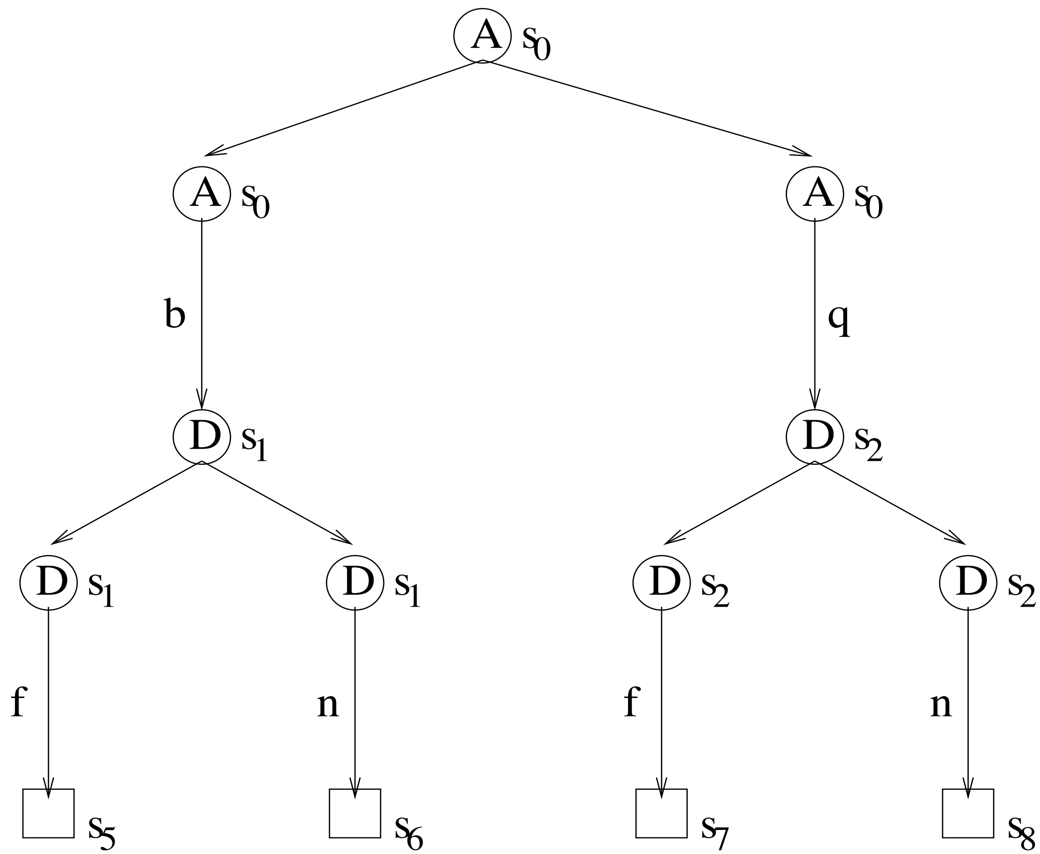


Let W mean that Al is strong. So,

$$\mathsf{W}^{\mathcal{I}} = \{s_6, s_8, s_{11}, s_{12}\}$$

The original scenario described in the example is that Al chooses whether or not to drink a beer, then Dick chooses whether or not to fight.

$$(b \cup q); (f \cap n)$$


A s_0
A s_0
A s_0
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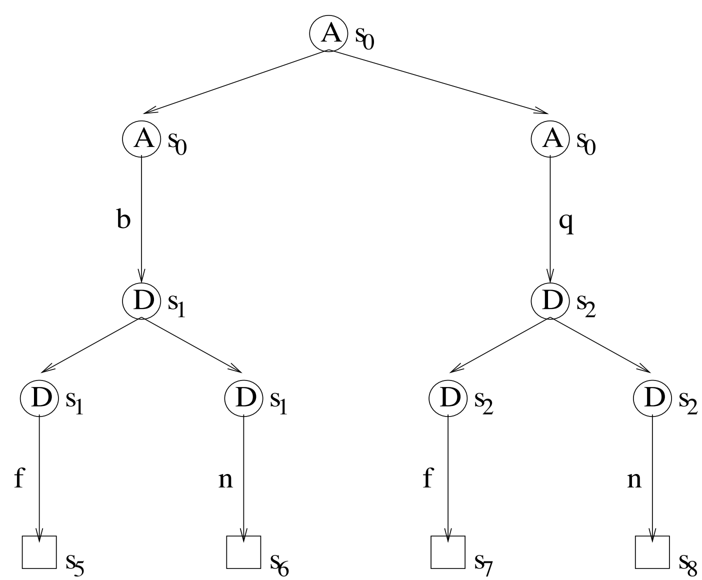
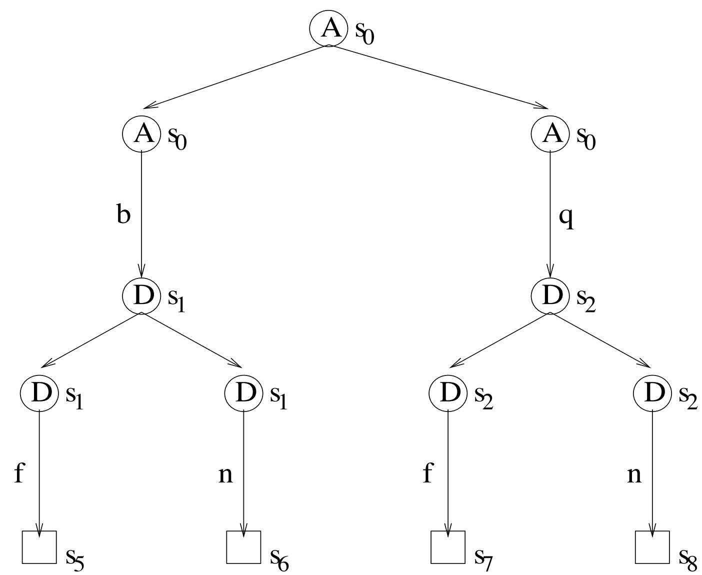
s_5
s_6
s_7
s_8



$\mathsf{S}^{\mathcal{I}} = \{s_6, s_7, s_9, s_{10}\}$

If Al is strong (Dick knows Al is strong), then Dick has a winning strategy

$$\mathcal{I}, s_0 \models [(b \cup q); (f \cap n)]\neg\mathsf{S}$$



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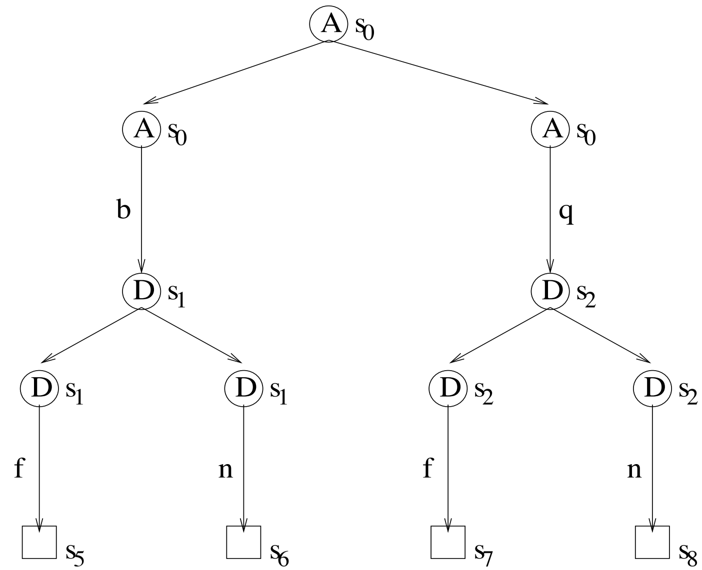
If Al is weak (Dick knows AL is weak), then Dick has a winning strategy

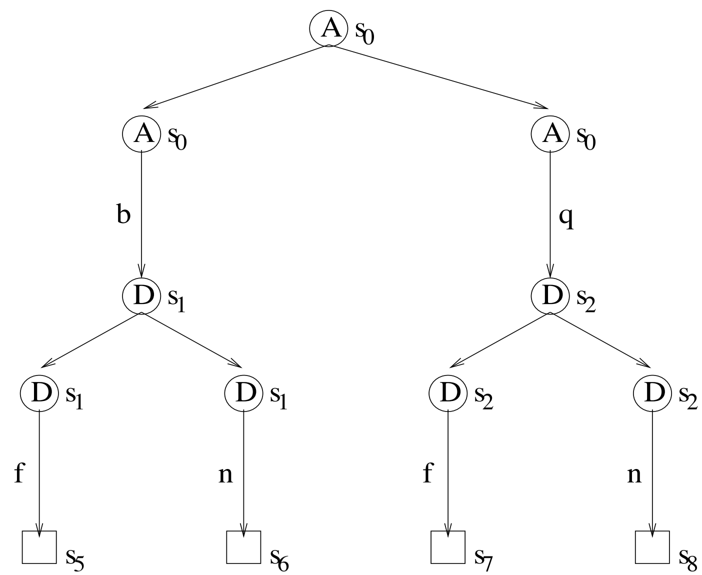
$$\mathcal{I}, s_0 \models [(b \cup q); (f \cap n)]\neg\mathsf{W}$$



$\mathsf{W}^{\mathcal{I}} = \{s_6, s_8, s_{11}, s_{12}\} \quad \mathsf{S}^{\mathcal{I}} = \{s_6, s_7, s_9, s_{10}\}$

Dick cannot win if he does not know whether Al is weak or strong

$$\mathcal{I}, s_0 \not\models [(b \cup q); (f \cap n)](\neg\mathsf{W} \wedge \neg\mathsf{S})$$



$$\mathsf{W}^{\mathcal{I}} = \{s_6, s_8, s_{11}, s_{12}\} \qquad \mathsf{S}^{\mathcal{I}} = \{s_6, s_7, s_9, s_{10}\}$$

Al has no winning strategy whether he is weak or strong

$$\mathcal{I}, s_0 \models \neg\langle (b \cup q); (f \cap n)\rangle \mathsf{W} \wedge \neg\langle (b \cup q); (f \cap n)\rangle \mathsf{S}$$

Theorem The two different semantics for game logic are equivalent.

M. Pauly. *Game Logic for Game Theorists.* available at the author's website.