Towards a Theory of Correctness of Social Procedures

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Introduction: Social Software

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For more information see

R. Parikh. *Social Software*. *Synthese* **132** (2002).

R. Parikh. *Language as Social Software*. in *Future Pasts: The Analytic Tradition in the Twentieth Century* (2001).

EP and R. Parikh. *Social Interaction, Knowledge, and Social Software*. in *Interactive Computation: The New Paradigm* (forthcoming).

Logic for Mechanism Design

Computational aspects of computer science vs. using ideas from computer science (eg. program verification) in game theory.

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Formally verifying mechanisms:

J. Halpern. *A Computer Scientist Looks at Game Theory. Games and Economic Behavior* **45** (2003).

J. van Benthem. *Extensive Games as Process Models. JOLLI* **11** (2002).

M. Pauly and M. Wooldridge. *Logics for Mechanism Design — A Manifesto.* available at the author's websites.

S. van Otterloo. *Strategic Analysis of Multi-agent Protocols.* Ph.D. Thesis, University of Liverpool (2005).

Outline of the Talk

- Strategy Logics
  - Coalitional Logic
    - A Simple Example
  - Alternating Time Temporal Logic
- From Hoare Logic to **PDL**
- Game Logic
  - Banach-Knaster Cake Cutting Algorithm
- Pauly's Mechanism Programming Language
  - Example
- Case Study: Adjusted Winner

From Temporal Logic to Strategy Logic

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- *Linear Time Temporal Logic*: Reasoning about computation paths:

$$\Diamond\phi\text{: } \phi \text{ is true some time in } \textit{the} \text{ future.}$$

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A. Pnuelli. *A Temporal Logic of Programs.* in *Proc. 18th IEEE Symposium on Foundations of Computer Science* (1977).

- *Branching Time Temporal Logic*: Allows quantification over paths:

  $\exists\Diamond\phi$: there is a path in which ϕ is eventually true.

E. M. Clarke and E. A. Emerson. *Design and Synthesis of Synchronization Skeletons using Branching-time Temproal-logic Specifications.* In *Proceedings Workshop on Logic of Programs*, LNCS (1981).

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- *Alternating-time Temporal Logic*: Reasoning about (local and global) group power:

  $\langle\langle A \rangle\rangle \Box \phi$: The coalition A has a joint strategy to ensure that ϕ will remain true.

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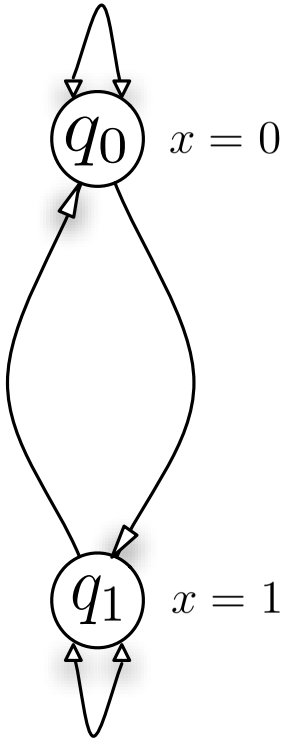
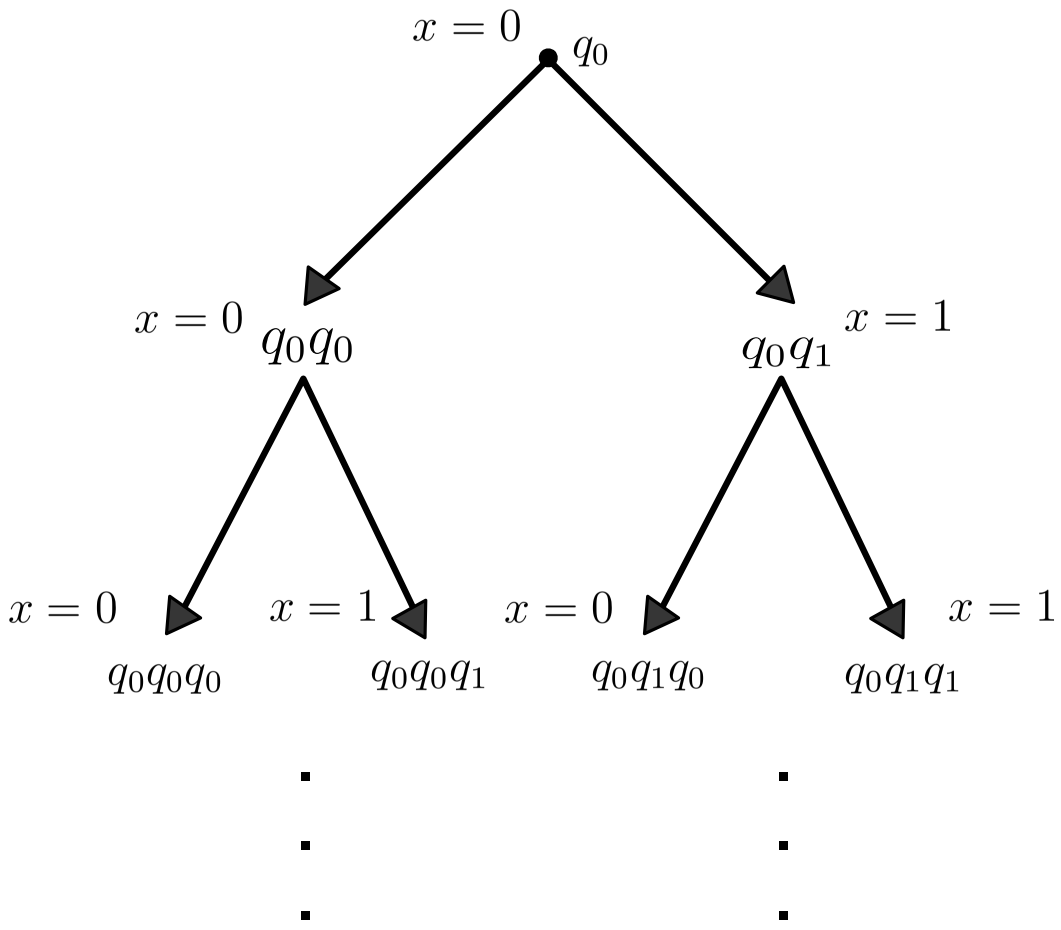
- *Coalitional Logic*: Reasoning about (local) group power (fragment of **ATL**).

  $[C]\phi$ (equivalently $\langle\langle C \rangle\rangle \bigcirc \phi$): coalition C has a joint strategy to bring about ϕ.

M. Pauly. *A Modal Logic for Coalition Powers in Games. Journal of Logic and Computation* **12** (2002).

Computational vs. Behavioral Structures





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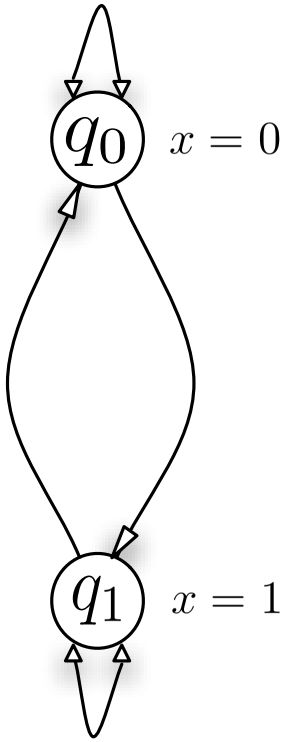

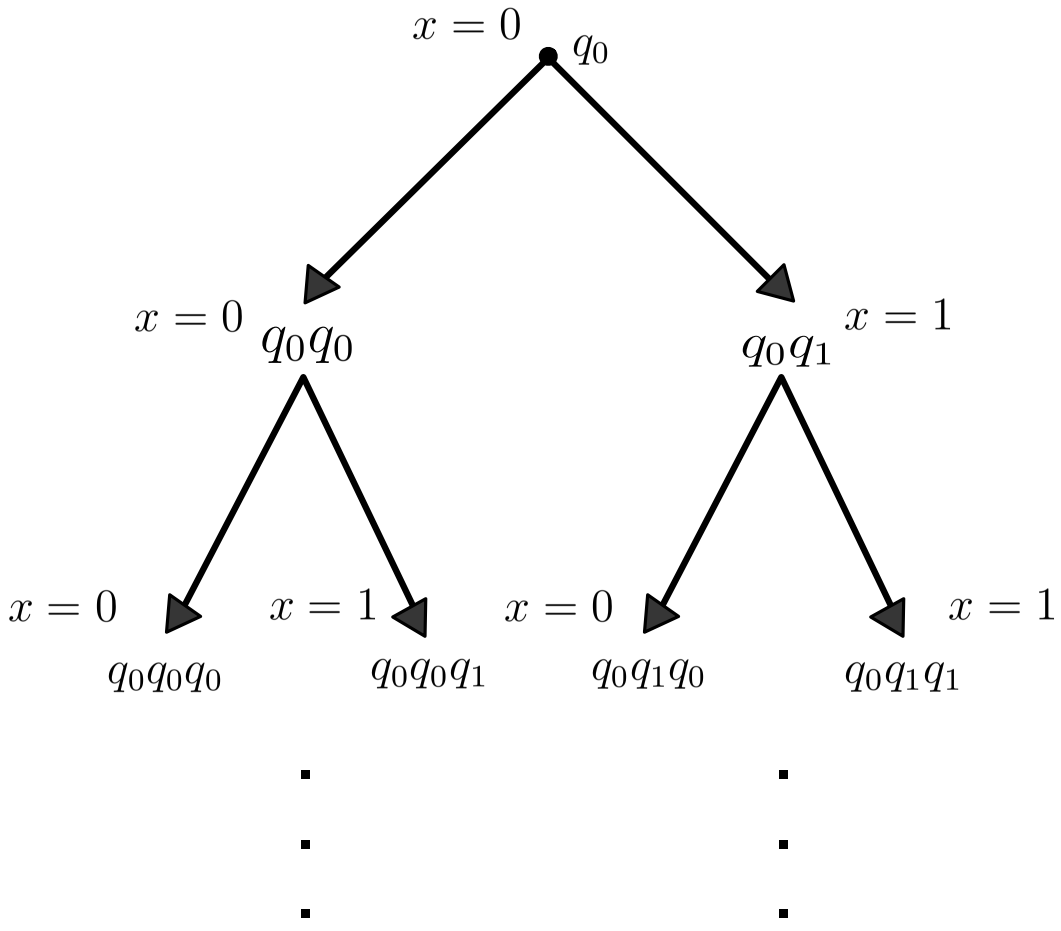

$$\exists \Diamond P_{x=1}$$

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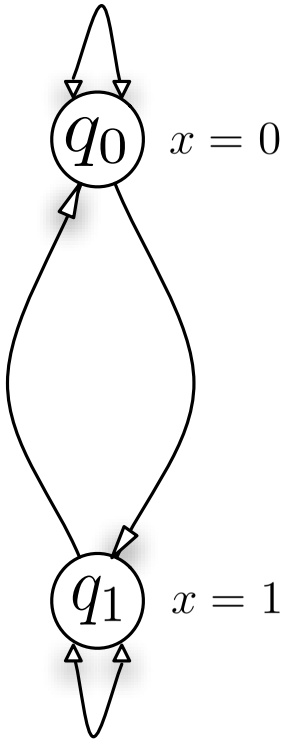

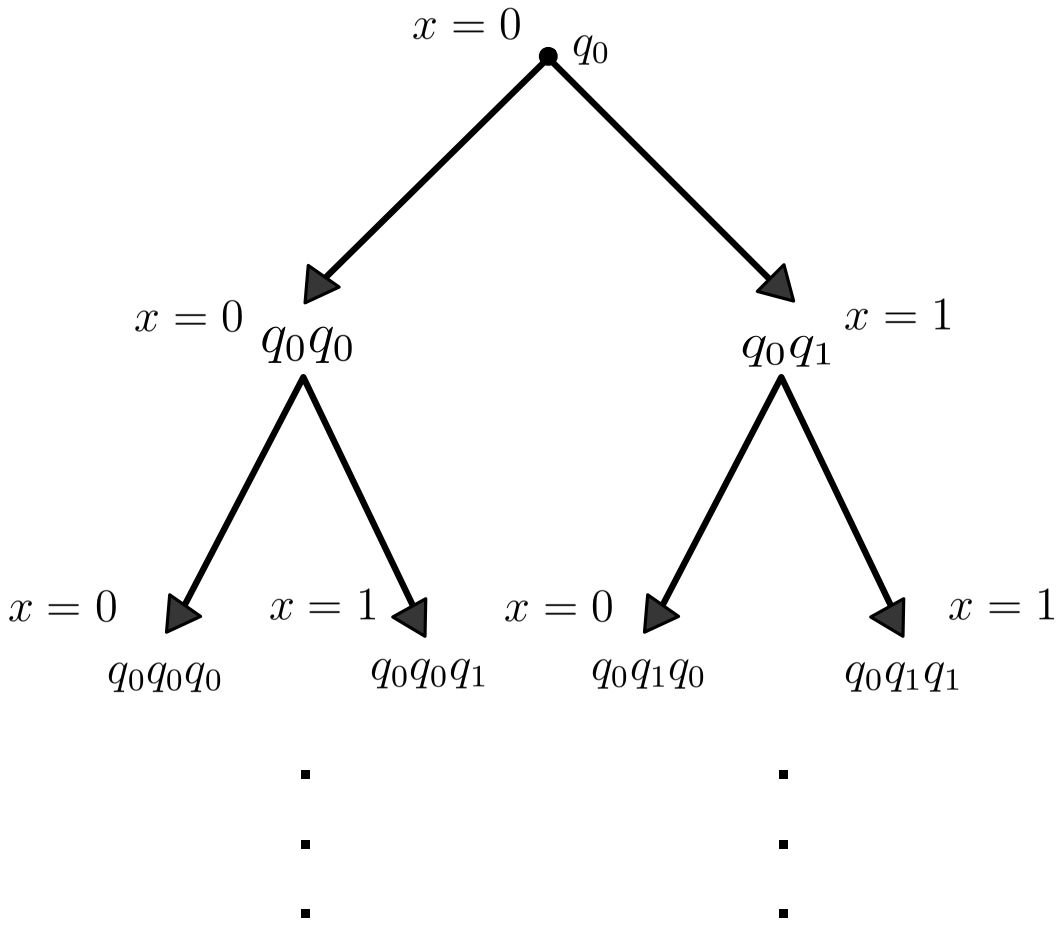

$$\neg\forall\Diamond P_{x=1}$$

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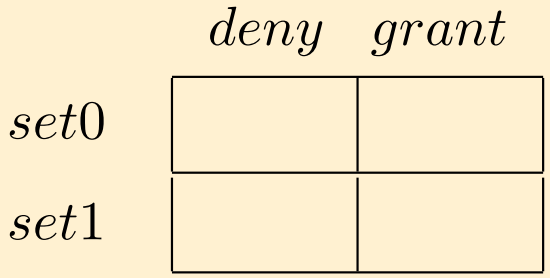

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*set*0		$q_0 \Rightarrow q_0,\ q_1 \Rightarrow q_0$
*set*1		$q_0 \Rightarrow q_1,\ q_1 \Rightarrow q_1$

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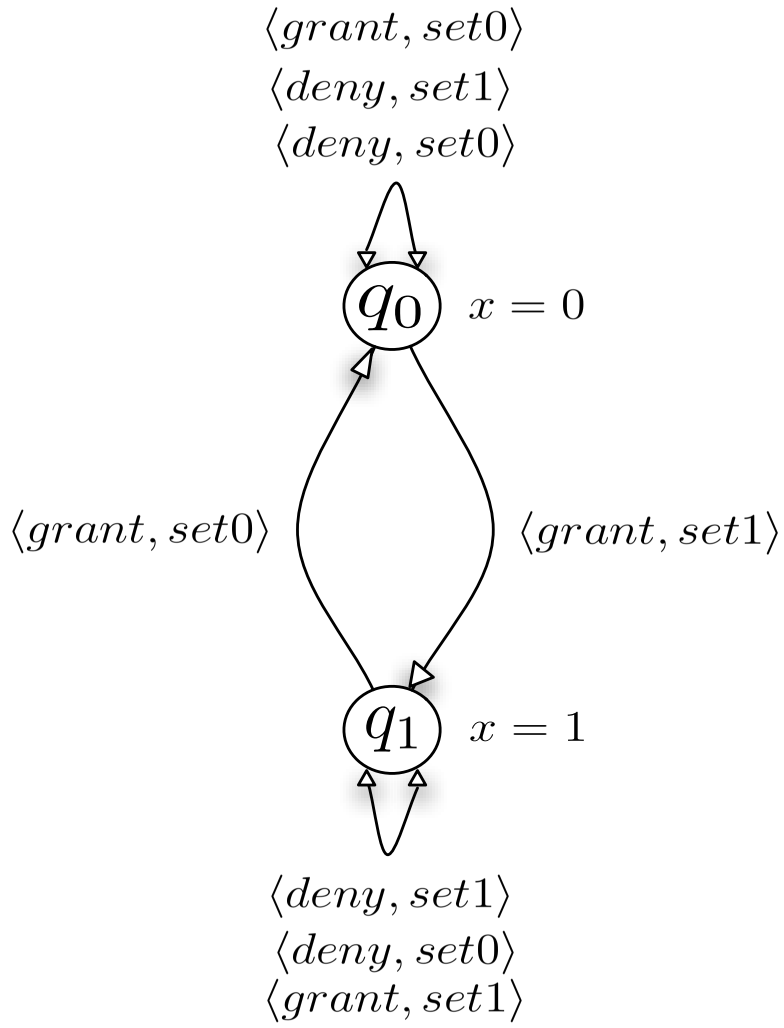
	deny	*grant*
$set0$	$q \Rightarrow q$	$q_0 \Rightarrow q_0$, $q_1 \Rightarrow q_0$
$set1$	$q \Rightarrow q$	$q_0 \Rightarrow q_1$, $q_1 \Rightarrow q_1$

Multi-agent Transition Systems



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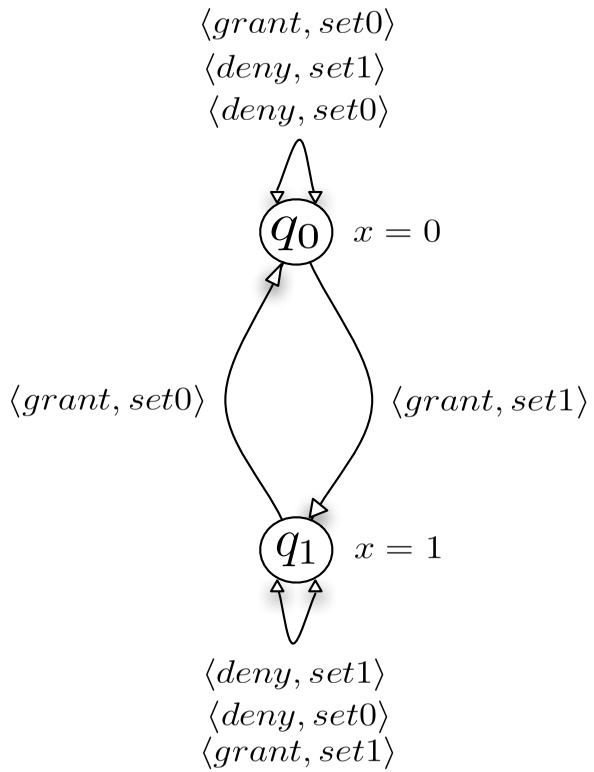

$$(P_{x=0} \rightarrow [s]P_{x=0}) \wedge (P_{x=1} \rightarrow [s]P_{x=1})$$

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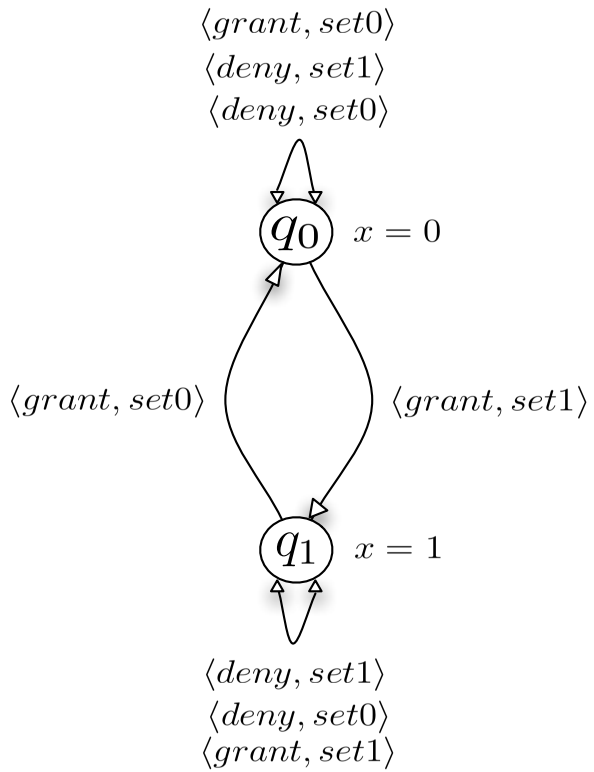

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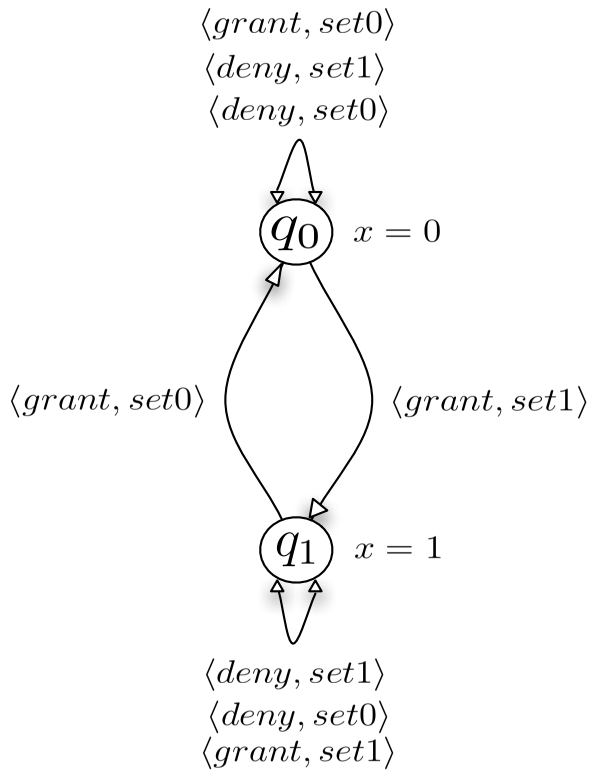

$$P_{x=0} \rightarrow [s, c]P_{x=1}$$

An Example

Two agents, A and B, must choose between two outcomes, p and q. We want a mechanism that will allow them to choose, which will satisfy the following requirements:

1. We definitely want an outcome to result, i.e., either p or q must be selected
2. We want the agents to be able to collectively choose and outcome
3. We do not want them to be able to bring about both outcomes simultaneously
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3. We do not want them to be able to bring about both outcomes simultaneously: $\neg[A, B](p \wedge q)$
4. We want them both to have equal power: $\neg[x]p \wedge \neg[x]q$ where $x \in \{A, B\}$

An Example

Consider the following mechanism:

The two agents vote on the outcomes, i.e., they choose either p or q. If there is a consensus, then the consensus is selected; if there is no consensus, then an outcome p or q is selected non-deterministically.

Pauly and Wooldridge use the MOCHA model checking system to verify that the above procedure satisfies the previous specifications.

Pauly's Coalitional Logic: Syntax

Given a finite non-empty set of agents N and a set of atomic propositions Φ_0, a formula ϕ can have the following syntactic form

$$\phi \;:=\; \bot \mid p \mid \neg\phi \mid \phi \vee \phi \mid [C]\phi$$

where $p \in \Phi_0$ and $C \subseteq N$.

$[C]\phi$ is intended to mean "coalition C can (locally) force ϕ to be true"

M. Pauly. *Logics for Social Software*. Ph.D. Thesis, ILLC (2001).

Multi-player Game Models

A **Strategic Game Form** is a tuple $\langle N, \{\Sigma_i \mid i \in N\}, Q, o\langle$ where

- N is a set of agents
- Σ_i is a set of actions
- Q is a set of states
- $o : \Pi_{i \in N} \Sigma_i \rightarrow Q$ assigns an outcome to each choice of aciton.

Let Γ_Q^N be the set of all such strategic game forms.

A **Multi-Player Game Model** is a tuple $\langle Q, \gamma, \pi \rangle$ where Q is a set of states and $\gamma : Q \rightarrow \Gamma_Q^N$ associates strategic games form to each state

$$q \models [C]\phi \text{iff } \exists\ \sigma_C\ \forall\ \sigma_{N-C},\ o(\sigma_C, \sigma_{N-C}) \models \phi$$

Effectivity Functions

Let G be a strategic game.

$$X \in E^{\alpha}_{G}(C) \text{iff } \exists \sigma_C \forall \sigma_{\overline{C}} \;\; o(\sigma_C, \sigma_{\overline{C}}) \in X$$

$$X \in E^{\beta}_{G}(C) \text{iff } \forall \sigma_{\overline{C}} \exists \sigma_C \;\; o(\sigma_C, \sigma_{\overline{C}}) \in X$$

$E^{\alpha}_{G} \subseteq E^{\beta}_{G}$

$$E_G^\beta \not\subseteq E_G^\alpha$$

Player 1 chooses the row, Player 2 chooses the column, Player 3 chooses the table

	l	m	r
l	s_1	s_2	s_1
r	s_2	s_1	s_3

	l	m	r
l	s_3	s_1	s_2
r	s_2	s_3	s_3

$$\{s_2\} \in E_G^\beta(\{2\}) \text{ but } \{s_2\} \notin E_G^\alpha(\{2\})$$

Coalition Effectivity Models

An effectivity function is **playable** iff

1. For each $C \subseteq N$, $\varnothing \notin E(C)$
2. For each $C \subseteq N$, $Q \in E(C)$
3. If $X \notin E(N)$, then $Q - X \in E(\varnothing)$
4. If $X \subseteq Y$ and $X \in E(C)$ then $Y \in E(C)$
5. for all $C_1, C_2 \subseteq N$ and $X_1, X_2 \subseteq Q$, if $C_1 \cap C_2 = \varnothing$, $X_1 \in E(C_1)$ and $X_2 \in E(C_2)$ then $X_1 \cap X_2 \in E(C_1 \cup C_2)$

Characterization Theorem: An α-effectivity function E is playable iff it is the effectivity function of some strategic game.

M. Pauly. *A Modal Logic for Coalition Powers in Games. Journal of Logic and Computation* **12** (2002).

Coalitional Logic: Coalition Effectivity Models

A **coalitional effectivity model** is a tuple $\langle Q, E, V \rangle$ where $E : Q \to (2^N \to 2^{2^Q})$ assigns a playable effectivity function to each state and V is a valuation function.

$$q \models [C]\phi \text{ iff } (\phi)^{\mathcal{M}} \in E_q(C)$$

Main Results

Theorem Coalitional Logic is sound and strongly complete with respect to the class of effectivity models.

Theorem The complexity of the satisfiability problem of coalitional logic is PSPACE-complete.

M. Pauly. *A Modal Logic for Coalitional Powers in Games. Journal of Logic and Computation* (2002).

M. Pauly. *On the Complexity of Coalitional Reasoning . International Game Theory Review* (2002).

ATL: Syntax

Let $\mathcal{A}$ be a set of agents, Π a set of propositional variables and $A \subseteq \mathcal{A}$.

1. p where $p \in \Pi$
2. $\neg\phi$
3. $\phi \vee \psi$
4. $\langle\langle A \rangle\rangle \bigcirc \phi$ meaning 'The coalition A can force in the next move an outcome satisfying ϕ'
5. $\langle\langle A \rangle\rangle \Box \phi$ meaning 'The coalition A can maintain forever outcomes satisfying ϕ'
6. $\langle\langle A \rangle\rangle \phi U \psi$ meaning 'The coalition A can eventually force an outcome satisfying ψ while meanwhile maintaining the truth of ϕ

Coalition Logic is a Fragment of ATL

Define $[A]\phi$ to be $\langle\langle A\rangle\rangle \bigcirc \phi$

Multi-player Game Models and Concurrent-game Models only differ in notation

Coalitional Effectivity Models can be used as a semantics for **ATL**

Goranko and Jamroga. *Comparing Semantics of Logics fro Multi-Agent Systems*. See the website.

Results

Theorem All of the semantics (concurrent game structures, alternating transitions systems and coalitional effectivity models) are equivalent.

Goranko and Jamroga. *Comparing Semantics of Logics for Multi-Agent Systems.* See the website.

Theorem **ATL** is sound and (weakly) complete.

Theorem Given a finite set of players, the satisfiability problem for **ATL**-formulas over N with respect to concurrent game structures over N is EXPTIME-complete.

Goranko and van Drimmelen. *Complete Axiomatization and Decidability of the Alternating-Time Temporal Logic.* Theoretical Computer Science (2005).

From Hoare Logic to Game Logic

- *Hoare Logic* Partial Correctness of Procedures

  $\{\phi\}\alpha\{\psi\}$: If the program α begins in a state in which ϕ is true, then after α terminates (!), ψ will be true.

C. A. R. Hoare. *An Axiomatic Basis for Computer Programming.*. Comm. Assoc. Comput. Mach. 1969.

- *Propositional Dynamic Logic* (**PDL**) [Pratt, 1976]: Reason about programs explicitly:

  $[\alpha]\phi$: after executing α, ϕ is true.

C. A. R. Hoare. *An Axiomatic Basis for Computer Programming.*. Comm. Assoc. Comput. Mach. 1969.

From Hoare Logic to Game Logic

- *Game Logic* (**GL**) [Parikh, 1985]: Reasoning about games:

  $(\gamma)\phi$: Agent I has a strategy to bring about ϕ in game γ.

R. Parikh. *The Logic of Games and its Applications.*. Annals of Discrete Mathematics. (1985) .

- More information:

K.R. Apt and E.R. Olderog. Verification of Sequential and Concurrent Programs, *Second Edition* . Springer-Verlag (1997).

D. Harel, D. Kozen and J. Tiuryn. Dynamic Logic. MIT Press (2000).

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Motivation: Formally verify the "correctness" of a program via *partial correctness assertions*:

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While Rule: $$\frac{\{\phi \wedge \sigma\}\ \alpha\ \{\phi\}}{\{\phi\}\ \textbf{while}\ \sigma\ \textbf{do}\ \alpha\ \{\phi \wedge \neg\sigma\}}$$

Example: Euclid's Algorithm

$x := u;$

$y := v;$

`while` $x \neq y$ `do`

    `if` $x < y$ `then`

        $y := y - x;$

    `else`

        $x := x - y;$

Let $\phi := \gcd(x, y) = \gcd(u, v)$

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Then $\{\gcd(x, y) = \gcd(u, v)\}\ \alpha\ \{\gcd(x, y) = \gcd(u, v)\}$

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Hence by the `while`-rule (using a "weakening rule")

$$\frac{\{(\gcd(x,y) = \gcd(u,v)) \wedge (x \neq y)\}\ \alpha\ \{\gcd(x,y) = \gcd(u,v))\}}{\{\gcd(x,y) = \gcd(u,v)\}\ \textbf{while}\ \sigma\ \textbf{do}\ \alpha\ \{(\gcd(x,y) = \gcd(u,v)) \wedge \neg(x \neq y)\}}$$

Background: Propositional Dynamic Logic

Let P be a set of atomic programs and At a set of atomic propositions.

Formulas of **PDL** have the following syntactic form:

$$\phi := p \mid \bot \mid \neg\phi \mid \phi \vee \psi \mid [\alpha]\phi$$

$$\alpha := a \mid \alpha \cup \beta \mid \alpha; \beta \mid \alpha^* \mid \phi?$$

where $p \in \mathsf{At}$ and $a \in \mathsf{P}$.

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$\{\phi\}\ \alpha\ \{\psi\}$ **is replaced with** $\phi \rightarrow [\alpha]\psi$

From PDL to Game Logic

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Main Idea:

In **PDL**: $w \models \langle \pi \rangle \phi$: there is a run of the program π starting in state w that ends in a state where ϕ is true.

The programs in **PDL** can be thought of as *single player games*.

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Game Logic generalized **PDL** by considering two players:

In **GL**: $w \models \langle \gamma \rangle \phi$: Angel has a **strategy** in the game γ to ensure that the game ends in a state where ϕ is true.

From PDL to Game Logic

Consequences of two players:

From PDL to Game Logic

Consequences of two players:

$\langle\gamma\rangle\phi$: Angel has a strategy in γ to ensure ϕ is true

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From PDL to Game Logic

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- $\gamma_1 \cap \gamma_2 := (\gamma_1^d \cup \gamma_2^d)^d$: Demon chooses between γ_1 and γ_2
- $\gamma^x := ((\gamma^d)^*)^d$: Demon can choose how often to play γ (possibly not at all); each time he has played γ, he can decide whether to play it again or not.

Syntax

Let Γ_0 be a set of atomic games and At a set of atomic propositions. Then formulas of Game Logic are defined inductively as follows:

$$\gamma \;:=\; g \mid \phi? \mid \gamma;\gamma \mid \gamma \cup \gamma \mid \gamma^* \mid \gamma^d$$

$$\phi \;:=\; \bot \mid p \mid \neg\phi \mid \phi \vee \phi \mid \langle\gamma\rangle\phi \mid [\gamma]\phi$$

where $p \in \mathsf{At}, g \in \Gamma_0$.

Game Logic: Semantics I

A **neighborhood game model** is a tuple
$\mathcal{M} = \langle W, \{E_g \mid g \in \Gamma_0\}, V\rangle$ where

W is a nonempty set of states

For each $g \in \Gamma_0$, $E_g : W \to 2^{2^W}$ is an **effectivity function** such that if $X \subseteq X'$ and $X \in E_g(w)$ then $X' \in E_g(w)$.

$X \in E_g(w)$ means in state s, Angel has a strategy to force the game to end in *some* state in X (we may write wE_gX)

$V : \mathsf{At} \to 2^W$ is a valuation function.

Game Logic: Semantics

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Suppose $E_\gamma(Y) = \{s \mid Y \in E_g(s)\}$

- $E_{\gamma_1;\gamma_2}(Y) := E_{\gamma_1}(E_{\gamma_2}(Y))$
- $E_{\gamma_1 \cup \gamma_2}(Y) := E_{\gamma_1}(Y) \cup E_{\gamma_2}(Y)$
- $E_{\phi?}(Y) := (\phi)^{\mathcal{M}} \cap Y$
- $E_{\gamma^d}(Y) := \overline{E_\gamma(\overline{Y})}$
- $E_{\gamma^*}(Y) := \mu X.Y \cup E_\gamma(X)$

Some Results

Fact Game Logic is more expressive than **PDL**

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Theorem Game Logic^{-x}, where $x \in \{*, d\}$ is sound and complete with respect to the class of all game models.

Open Question Is (full) game logic complete with respect to the class of all game models?

R. Parikh. *The Logic of Games and its Applications.*. Annals of Discrete Mathematics. (1985) .

M. Pauly. *Logic for Social Software*. Ph.D. Thesis, University of Amsterdam (2001)..

Some Results

Theorem [2] Given a game logic formula ϕ and a finite game model $\mathcal{M}$, model checking can be done in time $O(|\mathcal{M}|^{ad(\phi)+1} \times |\phi|)$

Theorem [1,2] The satisfiability problem for game logic is in EXPTIME.

Theorem [1] Game logic can be translated into the modal μ-calculus

[1] R. Parikh. *The Logic of Games and its Applications.*. Annals of Discrete Mathematics. (1985) .

[2] M. Pauly. *Logic for Social Software.* Ph.D. Thesis, University of Amsterdam (2001)..

More Information

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Example: Banach-Knaster Cake Cutting Algorithm

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- The algorithm continues with $n-1$ participants.

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Correctness: The algorithm is "correct" iff each player has a winning strategy for achieving a fair outcome ($1/n$ of the pie according to p_i's own valuation).

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- $F(m, k) \rightarrow (c, i)(F(m, k-1) \wedge H(x))$

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Goal: *Derive* a formula expressing that every individual has a strategy that guarantees her fair share.

M. Pauly and R. Parikh. *Game Logic — An Overview.* Studia Logica **75**, 2003.

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A Hoare-style Logic for Reasoning about Mechanisms

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An **interpretation** $\mathcal{I}$ is a first order structure (a domain $\mathcal{D}_\mathcal{I}$ and an interpretation of function and relation symbols) and preference relations $\succeq_a^\mathcal{I}$ on $\mathcal{D}_\mathcal{I}$ for each agent a.

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Associate with each expression γ and state s a *semi-game* $G(\gamma, s, \mathcal{I})$

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A semi-game $G(\gamma, s, \mathcal{I})$ can be turned into a **game** by adding an outcome function $\hat{o}$ that assigns an element of $\mathcal{D}_{\mathcal{I}}$ to terminal histories.

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- A **predicate** is *any* set of states $P \subseteq \mathcal{S}_\mathcal{I}$
- An **e-predicate** is *any* subset $P \subseteq \mathcal{S}_\mathcal{I} \times \mathcal{D}_\mathcal{I}$

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Define strategies and strategy profiles (σ) as usual. Each strategy profile corresponds to a run $run(\sigma)$. Let s_σ denote the last state of (a finite) $run(\sigma)$.

Given an e-predicate Q, let

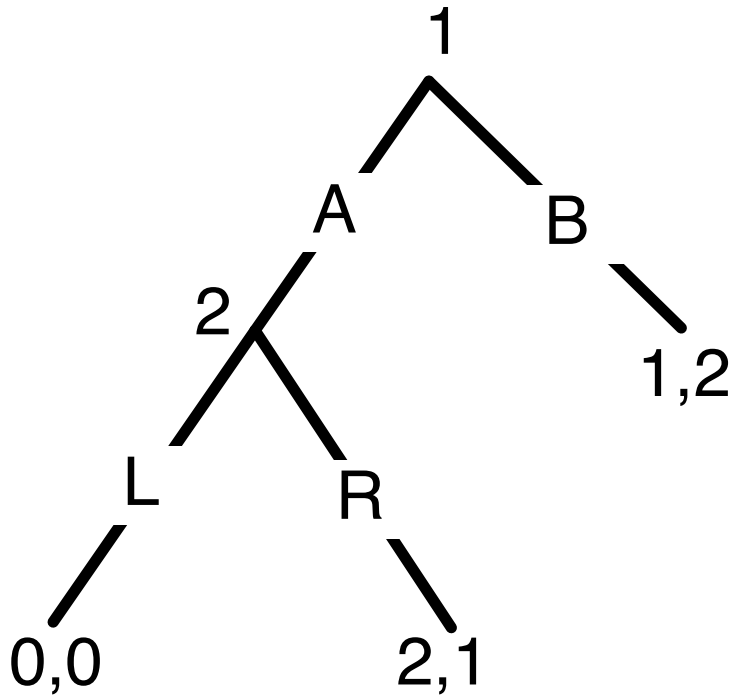
$\hat{O}_Q = \{\hat{o} \in \hat{O} \mid \text{for each terminal run } \sigma, \text{ if } \sigma \text{ is finite, then } (\mathsf{last}(\sigma), \hat{o}(\sigma)) \in Q\}$

Digression: Subgame Perfect Equilibrium

A **Nash Equilibrium** is a strategy profile in which no agent has an incentive to (unilaterally) deviate from their chosen strategy.

A **Subgame Perfect Equilibrium** is a strategy profile that is a Nash equilibrium in *every subgame*.

Digression: Subgame Perfect Equilibrium



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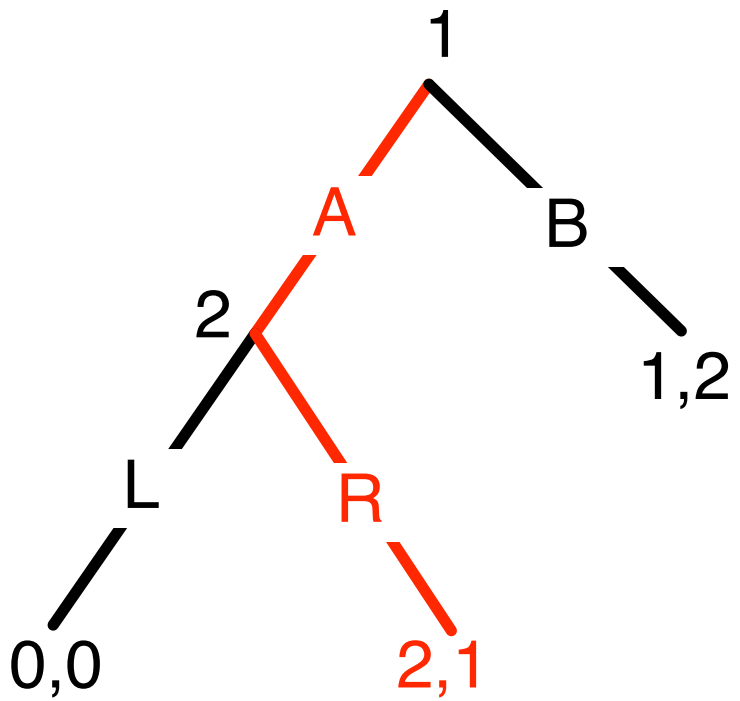

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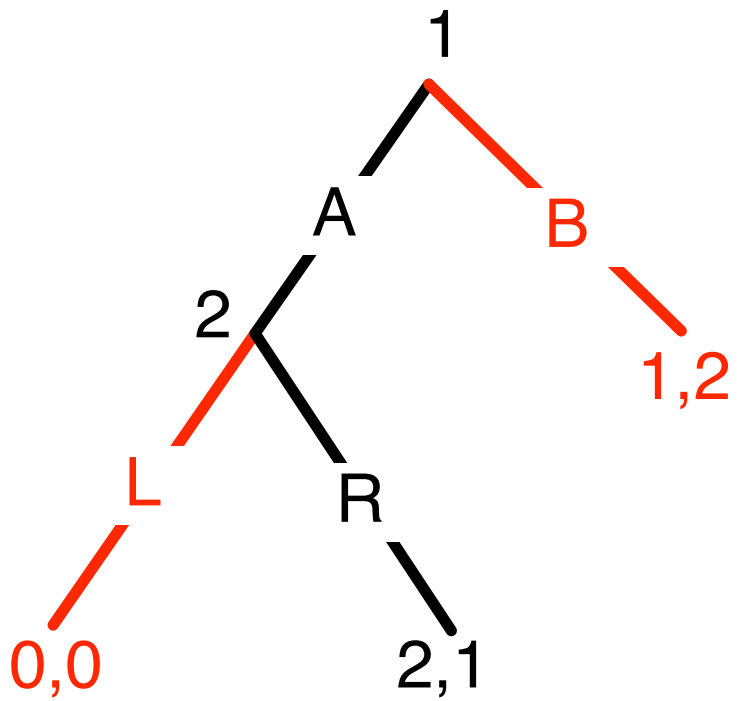

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A **correctness assertion** is an expression of the form $\{P\}\gamma\{Q\}$ where P, Q are **e-predicate**

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We say $\mathcal{I} \models \{P\}\gamma\{Q\}$ provided

> For each $(s, o) \in P$ there is a outcome function $\hat{f} \in \hat{O}_Q$ and a strategy profile σ such that σ is a **subgame perfect equilibrium** in $G(\gamma, s, \mathcal{I}, \hat{f})$ and $\hat{(}f)(s_\sigma) = o$.

Mechanism Design Problem

A **social choice correspondence** f maps a preference profile $(\succeq_i)_{i \in \mathcal{A}}$ to a set of outcomes $X \subseteq \mathcal{D}_{\mathcal{I}}$.

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Mechanism Design Problem: find a mechanism which implements the social choice correspondence such that no matter what the preferences of the agents are, self-interested agents will have an incentive to play so that the outcome intended by the designer will obtain.

M. Osborne and A. Rubinstein. A Course in Game Theory. Chapter 10.

Mechanism Design Problem

Give a social choice correspondence f, let
$f^*(x) = \{(s, o) \in S_{\mathcal{I}} \times \mathcal{D}_{\mathcal{I}} \mid o \in f(x)\}$ and let Q be any functional e-predicate.

Mechanism Design Problem

Give a social choice correspondence f, let
$f^*(x) = \{(s, o) \in S_{\mathcal{I}} \times \mathcal{D}_{\mathcal{I}} \mid o \in f(x)\}$ and let Q be any functional e-predicate.

Then we say that (γ, Q) SPE-implements a social choice correspondence f iff for all preference profiles $(\geq_i)_{i \in \mathcal{A}}$ we have

$$\mathcal{I}[(\geq_i)_{i \in \mathcal{A}}] \models \{f^*((\geq_i)_{i \in \mathcal{A}})\}\ \gamma\ \{Q\}$$

Solomon's Dilema

Two women have come before him with a small child, both claiming to be the mother of the child.

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Suppose that a is 'give the baby to A', b is 'give the baby to B' and c is 'cut the baby in half'.

Suppose

- $\Theta_1 : a >_1 b >_1 c$ and $b >_2 c >_2 a$
- $\Theta_2 : a >_1 c >_1 b$ and $b >_2 a >_2 c$

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- $\Theta_2 : a >_1 c >_1 b$ and $b >_2 a >_2 c$

Solomon must find a mechanism which implements the social choice rule $f(\Theta_1) = \{a\}$ and $f(\Theta_2) = \{b\}$.

It is well-known that f is not Nash-implementable

M. Osborne and A. Rubinstein. *A Course on Game Theory*. .

Solomon's Dilema

However, there is a solution involving money.

Allow Solomon to impose fines on the women, so outcomes are of the form:

$$(x, m_1, m_2)$$

where $x \in \{0, 1, 2\}$

Suppose the legitimate owner has valuation v_H and the other women has valuation v_L where

$$v_H > v_L > 0$$

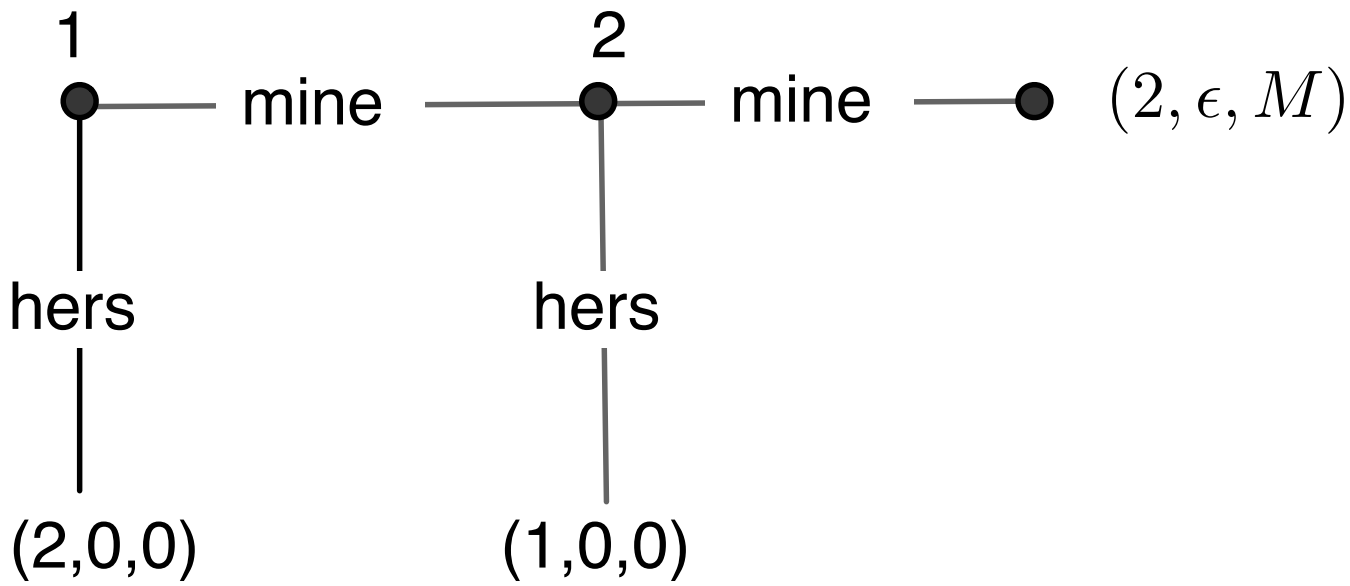
Solomon's Dilema

- If i does not get the painting then i's payoff is $-m_i$
- If i gets the painting and i is the legitimate owner then i's payoff is $v_H - m_i$
- If i gets the painting and i is not the legitimate owner then i's payoff is $v_L - m_i$

Solomon wishes to find a γ and Q such that $f(\Theta_i) = (i, 0, 0)$.

Let $\epsilon > 0$ and M be such that $v_L < M < v_H$.

Solomon's Dilema



Solomon's Dilema: A Formal Approach

$\mathsf{ch}_{\{1\}}(\{x_1\});$
$\texttt{if } x_1 > 0 \texttt{ then } owner := 2$
$\qquad \texttt{else } \mathsf{ch}_{\{2\}}(\{x_2\});$
$\qquad\qquad \texttt{if } x_2 > 0 \texttt{ then } owner := 1 \texttt{ else } owner := 0;$

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Q is the conjunction of

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Solomon's Dilema: A Formal Approach

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else $\mathsf{ch}_{\{2\}}(\{x_2\})$;
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$\mathcal{I}[\Theta_1] \models \{o = (1, 0, 0)\}\gamma\{Q\}$ and $\mathcal{I}[\Theta_2] \models \{o = (2, 0, 0)\}\gamma\{Q\}$

Adjusted Winner

Adjusted winner (AW) is an algorithm for dividing n divisible goods among two people (invented by Steven Brams and Alan Taylor).

For more information see

- *Fair Division: From cake-cutting to dispute resolution* by Brams and Taylor, 1998
- *The Win-Win Solution* by Brams and Taylor, 2000
- www.nyu.edu/projects/adjustedwinner

Adjusted Winner: Example

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Suppose Ann and Bob are dividing three goods: A, B, and C.

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Item	Ann	Bob
A	5	4
B	65	46
C	30	50
Total	100	100

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Item	Ann	Bob
A	5	0
B	65	0
C	0	50
Total	70	50

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Notice that $65/46 \geq 5/4 \geq 1 \geq 30/50$

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Suppose Ann and Bob are dividing three goods: A, B, and C.

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Still not equal, so give (some of) B to Bob: $65p = 100 - 46p$.

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yielding $p = 100/111 = 0.9009$

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Suppose Ann and Bob are dividing three goods: A, B, and C.

Step 3. Equitability adjustment:

yielding $p = 100/111 = 0.9009$

Item	Ann	Bob
A	0	4
B	58.559	4.559
C	0	50
Total	58.559	58.559

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Suppose that $G_1, \ldots, G_n$ is a fixed set of goods.

Adjusted Winner: Formal Definition

Suppose that $G_1, \ldots, G_n$ is a fixed set of goods.

A **valuation** of these goods is a vector of natural numbers $\langle a_1, \ldots, a_n \rangle$ whose sum is 100.

Let $\alpha, \alpha', \alpha'', \ldots$ denote possible valuations for Ann and $\beta, \beta', \beta'', \ldots$ denote possible valuations for Bob.

Adjusted Winner: Formal Definition

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An **allocation** is a vector of n real numbers where each component is between 0 and 1 (inclusive). An allocation $\sigma = \langle s_1, \ldots, s_n \rangle$ is interpreted as follows.

For each $i = 1, \ldots, n$, s_i is the proportion of G_i given to Ann.

Thus if there are three goods, then $\langle 1, 0.5, 0 \rangle$ means, "Give all of item 1 and half of item 2 to Ann and all of item 3 and half of item 2 to Bob."

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Adjusted Winner: Formal Definition

Suppose that $G_1, \ldots, G_n$ is a fixed set of goods.

$V_A(\alpha, \sigma) = \sum_{i=1}^{n} a_i s_i$ is the total number of points that Ann receives.

$V_B(\beta, \sigma) = \sum_{i=1}^{n} b_i(1 - s_i)$ is the total number of points that Bob receives.

Thus AW can be viewed as a function from pairs of valuations to allocations: $\mathsf{AW}(\alpha, \beta) = \sigma$ if σ is the allocation produced by the AW algorithm.

Adjusted Winner is Fair

Theorem *AW produces allocations that are efficient, equitable and envy-free (with respect to the announced valuations)*

S. Brams and A. Taylor. Fair Division. Cambridge University Press.

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$\mathsf{ch}_{\mathcal{A}}(\{x_1, x_2\})$;
$s := \mathrm{wta}(x_1, x_2)$;
`while` $\neg Eq(s, x_1, x_2)$ `do`
    $s := \mathrm{t}(s, x_1, x_2)$;

Adjusted Winner: Strategizing

Item	Ann	Bob
Matisse	75	25
Picasso	25	75

Ann will get the Matisse and Bob will get the Picasso and each gets 75 of his or her points.

Adjusted Winner: Strategizing

Suppose Ann knows Bob's preferences, but Bob does not know Ann's.

Item	Ann	Bob
M	75	25
P	25	75

Item	Ann	Bob
M	26	25
P	74	75

So Ann will get M plus a portion of P.

According to Ann's announced allocation, she receives 50 points

According to Ann's actual allocation, she receives
$75 + 0.33 * 25 = 83.33$ points.

Conclusion

- How should we deal with strategizing?

  EP. *Towards a Logical Analysis of Adjusted Winner.* working paper.

- Expressivity issues.

- Other equilibrium notions.

- Apply these ideas to more sophisticated mechanisms

Thank you.