Lecture 8 Handout

Knowledge and Common Knowledge

Definition 1 A **knowledge structure**[1] $\langle W, \{\sim_i\}_{i\in\mathcal{A}}\rangle$ where W is a nonempty set of states, $\sim_i \subseteq W \times W$ is an equivalence relation (*reflexive*: for each $w \in W$, $w \sim_i w$; *transitive*: for each $w, v, u \in W$, if $w \sim_i v$ and $v \sim_i u$ then $w \sim_i u$; and *symmetric*: for each $w, v \in W$, if $w \sim_i v$ then $v \sim_i w$). ◁

Remark 1 *It is sometimes convenient to work with* partitions *rather than equivalence relations. In this case a knowledge structure is a pair* $\langle W, \{\Pi_i\}_{i\in\mathcal{A}}\rangle$ *where each* Π_i *is a partition on* W*. A partition of* W *is a pairwise disjoint collection of subsets of* W *whose union is all of* W*. Elements of a partition* Π *on* W *are called* **cells***, and for* $w \in W$*, let* $\Pi(w)$ *denote the cell of* Π *containing* w*. There is a 1-1 correspondence between equivalence relations and partitions: Given an equivalence relation* $\sim_i$ *on* W*, the collection* $\Pi_i = \{[w]_i \mid w \in W\}$ *is a partition. Furthermore, given any partition* Π_i *on* W*,* $\sim_i = \{(w, v) \mid v \in \Pi_i(w)\}$ *is an equivalence relation with* $[w]_i = \Pi_i(w)$*.*

Given any knowledge structure $\langle W, \{\sim_i\}_{i\in\mathcal{A}}\rangle$ we define the following functions:

i ***knows*:** $K_i : \wp(W) \to \wp(W)$ is defined as $K_i(E) = \{w \mid [w]_i \subseteq E\}$

***everyone knows*:** $K : \wp(W) \to \wp(W)$ is defined as $K(E) = \bigcap_{i\in\mathcal{A}} K_i(E)$

***everyone knows … everyone knows that (*m*-times)*:** $K^m : \wp(W) \to \wp(W)$ is defined recursively as $K^0(E) = E$ and $K^{m+1}(E) = K(K^{m-1}(E))$

***common knowledge*:** $C : \wp(W) \to \wp(W)$ is defined as $C(E) = \bigcap_{m\geq 0} K^m(E)$

Note that we can rewrite $K_i(E)$ as follows $K_i(E) = \bigcup\{F \mid F \subseteq E, F \in \Pi_i\}$.

Fact 2 *Let* $\langle W, \{\sim_i\}_{i\in\mathcal{A}}\rangle$ *be an knowledge structure and define* $\sim_C \subseteq W \times W$ *as follows:* $\sim_C = (\bigcup_{i\in\mathcal{A}} \sim_i)^*$ *(for a relation* $R \subseteq W \times W$*,* R^* *is the smallest relation that is reflexive and transitive containing* R*). Then,* $w \in C(E)$ *iff for all* $v \in W$*, if* $w \sim_C v$ *then* $v \in E$*.*

Agreeing to Disagree: Non-Probabilistic Version

Definition 2 Let D be a nonempty set of **decisions**. A decision function for $i \in \mathcal{A}$ is a function $\mathbf{d}_i : W \to D$. A vector $\mathbf{d} = (\mathbf{d}_1, \ldots, \mathbf{d}_n)$ is a decision function profile. Define $[\mathbf{d}_i = d] = \{w \mid \mathbf{d}_i(w) = d\}$. ◁

[1] The terms "knowledge structure" or "epistemic structure" are used to describe a Kripke structure where the relations are equivalence relations.

Assumption 1 Each agent knows her own decision: for all decisions $d \in D$,

$$[\mathbf{d}_i = d] \subseteq K_i([\mathbf{d}_i = d])$$

Definition 3 Write $[j \succeq i]$ for the event *agent j is at least as knowledgeable as agent i*:

$$[j \succeq i] := \bigcap_{E \in \wp(W)} (K_i(E) \Rightarrow K_j(E)) = \bigcap_{E \in \wp(W)} (\neg K_i(E) \cup K_j(E))$$

Furthermore, define $[j \sim i] = [j \succeq i] \cap [i \succeq j]$ ◁

Then $w \in [j \succeq i]$ then j knows at w every event that i knows there.

Assumption 2 Interpersonal sure-thing principle (ISTP): for any pair of agents i and j and decision d, $K_i([j \succeq i] \cap [\mathbf{d}_j = d]) \subseteq [\mathbf{d}_i = d]$.

Lemma 3 *If the decision function profile* $\mathbf{d}$ *satisfies ISTP, then*

$$[i \sim j] \subseteq \bigcup_{d \in D} ([\mathbf{d}_i = d] \cap [\mathbf{d}_j = d])$$

Definition 4 Agent i is an **epistemic dummy** if it is always the case that all the agents are at least as knowledgeable as i. That is, for each agent j, $[j \succeq i] = W$. ◁

Definition 5 A decision function profile $\mathbf{d}$ on $\langle W, \{\sim_i\}_{i \in \mathcal{A}} \rangle$ is **ISTP expandable** if for any expanded structure $\langle W, \{\sim_i\}_{i \in \mathcal{A}} \cup \{\sim_{n+1}\} \rangle$ where $n+1$ is an epistemic dummy, there exists a decision function $\mathbf{d}_{n+1}$ such that $(\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_{n+1})$ satisfies ISTP. ◁

Theorem 4 (Agreeing to Disagree Theorem - Non-probabilistic Version) *If* $\mathbf{d}$ *is an ISTP expandable decision function profile on a knowledge structure* $\langle W, \{\sim_i\}_{i \in \mathcal{A}} \rangle$*, then for any decisions* $d_1, \ldots, d_n$ *which are not identical,* $C(\bigcap_{i \in \mathcal{A}} [\mathbf{d}_i = d_i]) = \emptyset$.

Bayesian Structures

Definition 6 A **Bayesian structure** is a tuple $\langle W, \{\sim_i\}_{i \in \mathcal{A}}, \{\pi_i\}_{i \in \mathcal{A}} \rangle$ where $\langle W, \{\sim_i\}_{i \in \mathcal{A}} \rangle$ is a knowledge structure and for each $i \in \mathcal{A}$, π_i is a probability measure on W. ◁

Remark 5 *For simplicity, we assume that W is finite. Then, we can define a probability for each subset $E \subseteq W$ as follows: $\pi_i(E) = \sum_{w \in E} \pi_i(w)$. Everything that follows holds when W is infinite, but we must take into details the usual issues that come with defining probabilities on an infinite set.*

Definition 7 For a probability measure π on W, the **conditional probability** of E given F is defined as follows $\pi(E \mid F) = \frac{\pi(E \cap F)}{\pi(F)}$. ◁

Definition 8 A Bayesian structure $\langle W, \{\sim_i\}_{i\in\mathcal{A}}, \{\pi_i\}_{i\in\mathcal{A}}\rangle$ has a **common prior** provided for all $i, j \in \mathcal{A}$, $\pi_i = \pi_j$. In such a case, we write π for π_i for all $i \in \mathcal{A}$. ◁

Definition 9 Given a Bayesian structure $\langle W, \{\sim_i\}_{i\in\mathcal{A}}, \{\pi_i\}_{i\in\mathcal{A}}\rangle$, the **posterior probability measure for agent** i, denoted $\overline{\pi_i} : W \to [0, 1]$, is defined as follows: $\overline{\pi_i}(w) = \pi_i(w \mid [w]_i)$. ◁

Remark 6 *Again, some care is needed here with the above definition. Recall that $\pi(E \mid F)$ is not defined if $\pi(F) = 0$, so the above definition implicitly assumes that for each $w \in W$, $\pi_i([w]_i) > 0$, which is a natural assumption.*

A second remark is that even if $\pi_i = \pi_j$ for agents i and j, we still have $\overline{\pi_i} \neq \overline{\pi_j}$ (since, in general, we have $[w]_i \neq [w]_j$).

Definition 10 Let $E \subseteq W$ and $r \in [0, 1]$. The event "the posterior probability for agent i assigned to event E is r", denoted $E_{i,r}$, is defined as follows $E_{i,r} = \{w \mid \overline{\pi_i}(E) = r_i\}$ ◁

Theorem 7 (Agreeing to Disagree) *Let $\langle W, \{\sim_i\}_{i\in\mathcal{A}}, \pi\rangle$ be a Bayesian structure with a common prior. Then for any event E and set of real numbers $\{p_1, \ldots, p_n\}$ with $p_i \in [0, 1]$ that are not all identical, $C(\bigcap_{i\in\mathcal{A}} E_{i,p_i}) = \emptyset$.*

Common p-Belief

Given any Bayesian structure $\langle W, \{\sim_i\}_{i\in\mathcal{A}}, \{\pi_i\}_{i\in\mathcal{A}}\rangle$, we define

> i ***believes to degree at least*** p**:** $B_i^p : \wp(W) \to \wp(W)$ is defined as $B_i^p(E) = \{w \mid \pi_i(E \mid [w]_i) \geq p\}$

Fact 8 *Let $\langle W, \{\sim_i\}_{i\in\mathcal{A}}, \pi\rangle$ be a Bayesian model with a common prior. Then,*

1. *$B_i^p(E)$ is the union of elements from Π_i.*
2. *If E is the union of elements from Π_i, then $B_i^p(E) = E$.*
3. *$B_i^p(B_i^p(E)) = B_i^p(E)$*
4. *If $E \subseteq F$ then $B_i^p(E) \subseteq B_i^p(F)$*
5. *If (E_n) is a decreasing sequence of events then*

$$B_i^p\left(\bigcap_n E_n\right) = \bigcap_n B_i^p(E_n)$$

6. *$\pi(E \mid B_i^p(E)) \geq p$*

Definition 11 An event E is an **evident p-belief** if for each $i \in \mathcal{A}$, $E \subseteq B_i^p(E)$. The common p-belief operator $C^p : \wp(W) \to \wp(W)$ is defined as follows:

$$C^p(F) = \{w \mid \text{ there is an evident } p\text{-belief } E \text{ with } w \in E \text{ and } E \subseteq B_i^p(F) \text{ for each } i \in \mathcal{A}\}$$

◁

Alternative definition of common p-belief. Given $\langle W, \{\sim_i\}_{i\in\mathcal{A}}, \{\pi_i\}_{i\in\mathcal{A}}\rangle$, we define

***everyone p-believes*:** $B^p : \wp(W) \to \wp(W)$ is defined as $B^p(E) = \bigcap_{i\in\mathcal{A}} B_i^p(E)$

***everyone knows* … *everyone knows that (m-times)*:** $B^{p(m)} : \wp(W) \to \wp(W)$ is defined recursively as $B^{p(0)}(E) = E$ and $B^{p(m+1)}(E) = B^p(B^{p(m-1)}(E))$

***common$_1$ p-belief*:** $C_1^p : \wp(W) \to \wp(W)$ is defined as $C_1^p(E) = \bigcap_{m\geq 1} B^{p(m)}(E)$

Proposition 9 *For each $E \subseteq W$, we have $C_1^p(E) = C^p(E)$.*

Theorem 10 (Generalized Agreeing to Disagree Theorem) *Let $\langle W, \{\sim_i\}_{i\in\mathcal{A}}, \pi\rangle$ be a Bayesian model with a common prior. For any event $E \subseteq W$, if $C^p\left(\bigcap_{i\in\mathcal{A}} E_{i,r_i}\right) \neq \emptyset$ then for each $i, j \in \mathcal{A}$, $|r_i - r_j| \leq 2(1-p)$.*