Logic and Artificial Intelligence

Lecture 15

Arthur Paul Pedersen

Department of Philosophy
Carnegie Mellon University

www.andrew.cmu.edu/∼ppederse
apaulpedersen@cmu.edu

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The Basic Problem of Belief Revision

Recall:

- Belief recognizes *belief, disbelief, and suspense.*
- Belief observes logical norms of logical closure and consistency.

We then ask:

Question: How ought a rational agent incorporate belief contravening information into a belief state?

Problem: Logical considerations alone are insufficient to answer this question!

So the really interesting question is:

What extralogical factors serve to determine what beliefs to give up and what beliefs to retain?

Quine's Two Dogmas

In his "Two Dogmas," Quine discusses *belief revision*.

Belief revision is a *matter of choice*, and the choices are to be made in such a way that:

(a) The resulting theory squares with the experience;
(b) It is simple; and
(c) The choices disturb the original theory as little as possible.

The Guiding Idea of research in belief revision :

(1) When accepting a new piece of information, an agent should aim at a minimal change of his old beliefs.
(2) If there are different ways to effect a belief change, the agent should give up those beliefs which are least entrenched.

Today

1. Review
2. Belief Contraction
3. Belief Revision
4. Propositional Models of Belief Change
5. Belief Change and Rational Choice?

Reference.

"Belief Revision." A.P. Pedersen & H. Arló-Costa. In L. Horsten and R. Pettigrew, editors, *Continuum Companion to Philosophical Logic*. Continuum Press, 2011.

Three Epistemic Changes

In the AGM framework, an agent's belief state is represented by a logically closed set of sentences K, called a *belief set*.

(i) In *expansion*, a sentence ϕ is added to a belief set K to obtain an expanded belief set $K + \phi$.

(ii) In *revision*, a sentence ϕ is added to a belief set K to obtain a revised belief set $K * \phi$ in a way that preserves logical consistency.

(iii) In contraction, a sentence ϕ is removed from K to obtain a contracted belief set $K \dot{-} \phi$ that does not include ϕ.

Revision can be reduced to contraction via the so-called *Levi identity*, according to which the revision of a belief set K with a sentence ϕ is identical to the contraction $K \dot{-} \neg\phi$ expanded by ϕ:

$$K * \phi \quad = \quad (K \dot{-} \neg\phi) + \phi.$$

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$$K * \phi \quad = \quad (K \dot{-} \neg\phi) + \phi.$$

Remainder Sets

Definition

Let K be a collection of formulae and α be a formula. The *α-remainder set* of K, $K \perp \alpha$, is the collection of subsets Γ of $\text{For}(\mathcal{L})$ such that:

(i) $\Gamma \subseteq K$;

(ii) $\alpha \notin \text{Cn}(\Gamma)$;

(iii) There is no set Δ such that $\Gamma \subset \Delta \subseteq K$ and $\alpha \notin \text{Cn}(\Delta)$.

A member of $K \perp \alpha$ is called an *α-remainder* of K. We let $K \perp \mathcal{L} := \{K \perp \alpha : \alpha \in \text{For}(\mathcal{L})\}$.

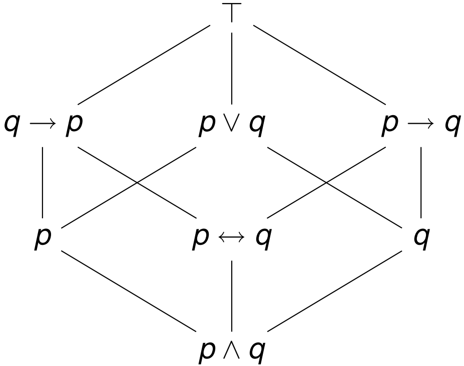
Example. Let $\mathcal{L} = \{p, q\}$, $\text{Cn} = \text{Cn}_0$, and $K = \text{Cn}(\{p, q\})$ Identify:

$K \perp (p \wedge q)$

$K \perp p$

$K \perp (q \rightarrow p)$

$$K\perp(p \wedge q) = \{\mathrm{Cn}(\{p \leftrightarrow q\}), \mathrm{Cn}(\{p\}), \mathrm{Cn}(\{q\})\}$$
$$K\perp p = \{\mathrm{Cn}(\{p \leftrightarrow q\}), \mathrm{Cn}(\{q\})\}$$
$$K\perp(q \rightarrow p) = \{\mathrm{Cn}(\{q\})\}.$$



Properties of Remainder Sets

Definition

Let K be a belief set and α be a formula. The α-*remainder set* of K, $K\perp\alpha$, is the collection of subsets Γ of $\mathrm{For}(\mathcal{L})$ such that:

(i) $\Gamma \subseteq K$;

(ii) $\alpha \notin \mathrm{Cn}(\Gamma)$;

(iii) There is no set Δ such that $\Gamma \subset \Delta \subseteq K$ and $\alpha \notin \mathrm{Cn}(\Delta)$.

A member of $K\perp\alpha$ is called an α-*remainder* of K. We let $K\perp\mathcal{L} := \{K\perp\alpha : \alpha \in \mathrm{For}(\mathcal{L})\}$.

Some important properties:

(a) $K\perp\alpha = \{K\}$ if and only if $\alpha \notin \mathrm{Cn}(K)$;

(b) $K\perp\alpha = \emptyset$ if and only if $\alpha \in \mathrm{Cn}(\emptyset)$.

The *Upper Bound Property*:

(c) If $\Gamma \subseteq K$ and $\alpha \notin \mathrm{Cn}(\Gamma)$, then there is some Δ such that $\Gamma \subseteq \Delta \in K\perp\alpha$.

Selection Functions

Definition

Let K be a belief set. A *selection function* for K is a function γ on $K \perp \mathcal{L}$ such that for all formulae α:

(i) If $K \perp \alpha \neq \emptyset$, then:
  (a) $\gamma(K \perp \alpha) \subseteq K \perp \alpha$, and
  (b) $\gamma(K \perp \alpha) \neq \emptyset$;

(ii) If $K \perp \alpha = \emptyset$, then $\gamma(K \perp \alpha) = \{K\}$.

Example.

$$K \perp (p \wedge q) = \{\mathrm{Cn}(\{p \leftrightarrow q\}), \mathrm{Cn}(\{p\}), \mathrm{Cn}(\{q\})\}$$
$$K \perp p = \{\mathrm{Cn}(\{p \leftrightarrow q\}), \mathrm{Cn}(\{q\})\}$$

Consider:

$$\gamma(K \perp (p \wedge q)) = \{\mathrm{Cn}(\{p\})\}$$
$$\gamma(K \perp p) = \{\mathrm{Cn}(\{p \leftrightarrow q\}), \mathrm{Cn}(\{q\})\}$$

Partial Meet Contraction

Definition

Let K be a set of formulae. A function $\dot{-}$ on For($\mathcal{L}$) is a *partial meet contraction* for K if there is a selection function γ for K such that for all formulae α:

$$K \dot{-} \alpha \quad = \quad \bigcap \gamma(K \perp \alpha).$$

Example.

$$K \perp (p \wedge q) = \{\mathrm{Cn}(\{p \leftrightarrow q\}), \mathrm{Cn}(\{p\}), \mathrm{Cn}(\{q\})\}$$
$$K \perp p = \{\mathrm{Cn}(\{p \leftrightarrow q\}), \mathrm{Cn}(\{q\})\}$$

Consider:

$$\gamma(K \perp (p \wedge q)) = \{\mathrm{Cn}(\{p\})\}$$
$$\gamma(K \perp p) = \{\mathrm{Cn}(\{p \leftrightarrow q\}), \mathrm{Cn}(\{q\})\}$$

So:

$$K \dot{-} (p \wedge q) = \bigcap \gamma(K \perp p \wedge q) = \mathrm{Cn}(\{p\})$$
$$K \dot{-} p = \gamma(K \perp p) = \mathrm{Cn}(\{q\})$$

Two limiting cases of partial meet contraction are of special interest:

(i) γ selects exactly one element of $K \perp \alpha$ (*maxichoice contraction*);
(ii) γ selects the entire set $K \perp \alpha$ (*full meet contraction*).

So we have:

(i) $K \dot{-} \alpha = \{\Gamma\}$ for some $\Gamma \in K \perp \alpha$;
(ii) $K \dot{-} \alpha = \bigcap K \perp \alpha$.

Contraction Postulates

The approach in question concerns itself with not only providing semantic characterizations of belief change but also supplying postulates belief formation operators ought to obey.

The primary *logical* goal of this approach is a representation result for a set of compelling rationality postulates.

$(K \dot{-} 1)$	$K \dot{-} \alpha = \mathrm{Cn}(K \dot{-} \alpha)$.	(*Closure*)
$(K \dot{-} 2)$	$K \dot{-} \alpha \subseteq K$.	(*Inclusion*)
$(K \dot{-} 3)$	If $\alpha \notin K$ or $\alpha \in \mathrm{Cn}(\emptyset)$, then $K \dot{-} \alpha = K$.	(*Vacuity*)
$(K \dot{-} 4)$	If $\alpha \notin \mathrm{Cn}(\emptyset)$, then $\alpha \notin K \dot{-} \alpha$.	(*Success*)
$(K \dot{-} 5)$	If $\mathrm{Cn}(\{\alpha\}) = \mathrm{Cn}(\{\beta\})$, then $K \dot{-} \alpha = K \dot{-} \beta$.	(*Extensionality*)
$(K \dot{-} 6)$	$K \subseteq \mathrm{Cn}((K \dot{-} \alpha) \cup \{\alpha\})$.	(*Recovery*)

These are the *basic* AGM postulates.

Recovery is the most controversial postulate from the foregoing list.

Problems with Recovery

Researchers have offered various counterexamples to Recovery.

Example (Hansson 1991)

While reading a book about Cleopatra I learned that she had both a son and a daughter. I therefore believe both that Cleopatra had a son (s) and Cleopatra had a daughter (d). Later I learn from a well-informed friend that the book in question is just a historical novel. I accordingly contract my belief that Cleopatra had a child ($s \vee d$). However, shortly thereafter I learn from a reliable source that in fact Cleopatra had a child. I thereby reintroduce $s \vee d$ to my collection of beliefs without also returning either s or d.

$$(K \dot{-} 6) \quad K \subseteq \mathrm{Cn}((K \dot{-} \alpha) \cup \{\alpha\}). \quad (\textit{Recovery})$$

Another proposed counterexample:

Example (Hansson 1996)

I believed both that George is a criminal (c) and George is a mass murderer (m). Upon receiving certain information I am induced to retract my belief set K by my belief that George is a criminal (c). Of course, I therefore retract my belief set by my belief that George is a mass murderer (m). Later I learn that in fact George is a shoplifter (s), so I expand my contracted belief set $K \dot{-} c$ by s to obtain $(K \dot{-} c) + s$. As George's being a shoplifter (s) entails his being a criminal (c), $(K \dot{-} c) + c$ is a subset of $(K \dot{-} c) + s$. Yet by Recovery it follows that $K \subseteq (K \dot{-} c) + c$, so m is a member of the expanded belief set $(K \dot{-} c) + s$. But I do not believe that George is a mass murdered (m).

$(K \dot{-} 6) \quad K \subseteq \mathrm{Cn}((K \dot{-} \alpha) \cup \{\alpha\}). \quad$ (*Recovery*)

While Gärdenfors (1982) contends that Recovery is a reasonable principle, Makinson expresses doubts about Recovery (1987) and at the same time defends its use in certain contexts (1997).

Makinson (1997) argues that the foregoing examples are persuasive only as a result of tacitly adding to the theory of contraction a *justificatory structure* that is not formally represented.

For example, Makinson claims that in the second example we are inclined to take for granted that $m \vee \neg s$ is in the belief set *only because* m is there. Makinson concludes:

> *As soon as contraction makes use of the notion 'y is believed only because x,' we run into counterexamples to recovery, like those of Cleopatra and [the shoplifter]. But when a theory is 'naked,' i.e. as a bare set* $A = \mathrm{Cn}(A)$ *of statements closed under consequence, then recovery appears to be free of intuitive counterexamples (Makinson 1997, p. 478).*

Thus Makinson seemingly argues that Recovery can fail only in cases in which some justificatory structure is added to the belief set and used to determine the content of a contraction.

Definition

Let K be a belief set. A function $\dot{-}$ on For($\mathcal{L}$) is a *relational partial meet contraction* for K if there is a selection function γ for K and a binary relation $\preceq$ on $K\perp\mathcal{L}$ such that for every formula α:

(i) $K \dot{-} \alpha = \bigcap \gamma(K\perp\alpha)$;

(ii) If $K\perp\alpha \neq \emptyset$, then $\gamma(K\perp\alpha) = \{\Gamma \in K\perp\alpha : \Lambda \preceq \Gamma \text{ for all } \Lambda \in K\perp\alpha\}$.

If such a relation $\preceq$ is in addition transitive, then we call such $\dot{-}$ *transitively relational*.

This semantic requirement is reflected in two *supplementary postulates*:

$(K \dot{-} 7)$	$(K \dot{-} \alpha) \cap (K \dot{-} \beta) \subseteq K \dot{-} (\alpha \wedge \beta)$.	(*Overlap*)
$(K \dot{-} 8)$	If $\alpha \notin K \dot{-} (\alpha \wedge \beta)$, then $K \dot{-} (\alpha \wedge \beta) \subseteq K \dot{-} \alpha$.	(*Inclusion*)

Representation Theorem

Recall the basic and supplementary postulates:

$(K \dot{-} 1)$	$K \dot{-} \alpha = \mathrm{Cn}(K \dot{-} \alpha)$.	(*Closure*)
$(K \dot{-} 2)$	$K \dot{-} \alpha \subseteq K$.	(*Inclusion*)
$(K \dot{-} 3)$	If $\alpha \notin K$ or $\alpha \in \mathrm{Cn}(\emptyset)$, then $K \dot{-} \alpha = K$.	(*Vacuity*)
$(K \dot{-} 4)$	If $\alpha \notin \mathrm{Cn}(\emptyset)$, then $\alpha \notin K \dot{-} \alpha$.	(*Success*)
$(K \dot{-} 5)$	If $\mathrm{Cn}(\{\alpha\}) = \mathrm{Cn}(\{\beta\})$, then $K \dot{-} \alpha = K \dot{-} \beta$.	(*Extensionality*)
$(K \dot{-} 6)$	$K \subseteq \mathrm{Cn}((K \dot{-} \alpha) \cup \{\alpha\})$.	(*Recovery*)
$(K \dot{-} 7)$	$(K \dot{-} \alpha) \cap (K \dot{-} \beta) \subseteq K \dot{-} (\alpha \wedge \beta)$.	(*Conjunctive Overlap*)
$(K \dot{-} 8)$	If $\alpha \notin K \dot{-} (\alpha \wedge \beta)$, then $K \dot{-} (\alpha \wedge \beta) \subseteq K \dot{-} \alpha$.	(*Conjunctive Inclusion*)

The centerpiece of AGM's influential 1985 paper:

Theorem (AGM 1985)

Let K be a belief set, and let $\dot{-}$ be a function on $\mathrm{For}(\mathcal{L})$. *Then:*

(i) *The function $\dot{-}$ is a partial meet contraction for K if and only if it satisfies postulates $(K \dot{-} 1)$ to $(K \dot{-} 6)$.*

(ii) *The function $\dot{-}$ is a transitively relational partial meet contraction for K if and only if it satisfies postulates $(K \dot{-} 1)$ to $(K \dot{-} 8)$.*

Definition

Let K be a set of formulae. A function $*$ on $\mathrm{For}(\mathcal{L})$ is a *partial meet revision* for K if there is a selection function γ for K such that for all formulae α:

$$K * \alpha \quad = \quad \mathrm{Cn}((\bigcap \gamma(K \perp \neg\alpha)) \cup \{\alpha\}).$$

The *basic* revision postulates are analogues of the basic contraction postulates:

$(K*1)$	$K * \phi = \mathrm{Cn}(K * \phi)$.	(*Closure*)
$(K*2)$	$\phi \in K * \phi$.	(*Success*)
$(K*3)$	$K * \phi \subseteq \mathrm{Cn}(K \cup \{\phi\})$.	(*Inclusion*)
$(K*4)$	If $\neg\phi \notin K$, then $\mathrm{Cn}(K \cup \{\phi\}) \subseteq K * \phi$.	(*Vacuity*)
$(K*5)$	If $\mathrm{Cn}(\{\phi\}) \neq \mathrm{For}(\mathcal{L})$, then $K * \phi \neq \mathrm{For}(\mathcal{L})$.	(*Consistency*)
$(K*6)$	If $\mathrm{Cn}(\{\phi\}) = \mathrm{Cn}(\{\psi\})$, then $K * \phi = K * \psi$.	(*Extensionality*)

Definition

Let K be a set of formulae. A function $*$ on For($\mathcal{L}$) is a *relational partial meet revision* for K if there is a selection function γ for K and a binary relation $\preceq$ on $K\perp\mathcal{L}$ such that for every formula α:

(i) $K * \alpha = \text{Cn}((\bigcap \gamma(K\perp\neg\alpha)) \cup \{\alpha\})$;

(ii) If $K\perp\alpha \neq \emptyset$, then $\gamma(K\perp\alpha) = \{\Gamma \in K\perp\alpha : \Lambda \preceq \Gamma \text{ for all } \Lambda \in K\perp\alpha\}$.

If such a relation $\preceq$ is in addition transitive, then we call such $*$ *transitively relational*.

As with contraction functions, the six basic postulates are elementary requirements of belief revision and taken by themselves are much too permissive.

Supplementary postulates rein in this permissiveness, reflecting the semantic notion of relational belief revision.

$(K*7)$ $\quad K * (\phi \wedge \psi) \subseteq \text{Cn}((K * \phi) \cup \{\psi\})$.

$(K*8)$ $\quad \neg\psi \notin K * \phi$, then $\text{Cn}(K * \phi \cup \{\psi\}) \subseteq K * (\phi \wedge \psi)$.

Recall the basic and supplementary postulates:

$(K*1)$	$K*\phi = \mathrm{Cn}(K*\phi)$.	(*Closure*)
$(K*2)$	$\phi \in K*\phi$.	(*Success*)
$(K*3)$	$K*\phi \subseteq \mathrm{Cn}(K \cup \{\phi\})$.	(*Inclusion*)
$(K*4)$	If $\neg\phi \notin K$, then $\mathrm{Cn}(K \cup \{\phi\}) \subseteq K*\phi$.	(*Vacuity*)
$(K*5)$	If $\mathrm{Cn}(\{\phi\}) \neq \mathrm{For}(\mathcal{L})$, then $K*\phi \neq \mathrm{For}(\mathcal{L})$.	(*Consistency*)
$(K*6)$	If $\mathrm{Cn}(\{\phi\}) = \mathrm{Cn}(\{\psi\})$, then $K*\phi = K*\psi$.	(*Extensionality*)
$(K*7)$	$K*(\phi \wedge \psi) \subseteq \mathrm{Cn}((K*\phi) \cup \{\psi\})$.	(*Superexpansion*)
$(K*8)$	$\neg\psi \notin K*\phi$, then $\mathrm{Cn}(K*\phi \cup \{\psi\}) \subseteq K*(\phi \wedge \psi)$.	(*Subexpansion*)

Theorem (AGM 1985)

Let K be a belief set, and let $*$ be a function on $\mathrm{For}(\mathcal{L})$. Then:

(i) *The function $*$ is a partial meet revision for K if and only if it satisfies postulates $(K*1)$ to $(K*6)$.*

(ii) *The function $*$ is a transitively relational partial meet revision for K if and only if it satisfies postulates $(K*1)$ to $(K*8)$.*

Propositional Models of Belief Change

The AGM framework for belief change uses the notion of a *remainder set* to define operators of belief change, so belief states and belief change have a syntactic character.

An alternative and arguably more suitable and elegant framework for belief change uses *propositions*, or sets of possible worlds, instead.

Propositional models of belief change can be connected to the syntactic models of belief change we have hereunto discussed, offering a useful visualization of the different operators of belief change.

See (Grove 1988).

Also: (Arló-Costa and Pedersen 2010), (Harper 1975, 1977), (Katsuno and Mendelzon 1989, 1991), (Pedersen 2008), (Rott 1993, 2001), (Spohn 1988, 1990, 1998).

Preliminaries

Let $\mathcal{W}_{\mathcal{L}}$ denote the collection of all maximal consistent sets of $\mathcal{L}$ with respect to Cn (*worlds* w).

For $A \subseteq \mathcal{W}_{\mathcal{L}}$, let $\text{Th}(A) := \bigcap A$ (if $A = \emptyset$, $\text{Th}(A) := \text{For}(\mathcal{L})$)

For $\Gamma \subseteq \text{For}(\mathcal{L})$, let $[\![\Gamma]\!] := \{w \in \mathcal{W}_{\mathcal{L}} : \Gamma \subseteq w\}$.

For $\phi \in \text{For}(\mathcal{L})$, write $[\![\phi]\!]$ instead of $[\![\{\phi\}]\!]$.

A member of $\mathscr{P}(\mathcal{W}_{\mathcal{L}})$ is often called a *proposition*, and $[\![\phi]\!]$ is often called the *proposition expressed by* ϕ.

Finally, let $\mathcal{E}_{\mathcal{L}} := \{A \in \mathscr{P}(\mathcal{W}_{\mathcal{L}}) : A = [\![\phi]\!] \text{ for some } \phi \in \text{For}(\mathcal{L})\}$ (elementary sets)

Sphere-Based Revision

Proposed by (Grove 1988), so-called *sphere semantics* offers an elegant representation of belief change.

Definition

Let $C \subseteq \mathcal{W}_{\mathcal{L}}$, and let $\mathscr{S} \subseteq \mathscr{P}(\mathcal{W}_{\mathcal{L}})$. We call $\mathscr{S}$ a *system of spheres centered on C* if it satisfies the following properties:

($\mathscr{S}$1) $\mathscr{S}$ is totally ordered by $\subseteq$;
($\mathscr{S}$2) C is the $\subseteq$-minimum of $\mathscr{S}$;
($\mathscr{S}$3) $\mathcal{W}_{\mathcal{L}} \in \mathscr{S}$;
($\mathscr{S}$4) For every formula ϕ and $S \in \mathscr{S}$, if $S \cap [\![\phi]\!] \neq \emptyset$, then there is a $\subseteq$-minimum $S_0 \in \mathscr{S}$ such that $S_0 \cap [\![\phi]\!] \neq \emptyset$.

Now for each formula ϕ, define the following set:

$$\mathscr{C}_\phi \quad := \quad \{S \in \mathscr{S} : S \cap [\![\phi]\!] \neq \emptyset\} \cup \{\mathcal{W}_{\mathcal{L}}\}.$$

Propositional Selection Functions

Independent of sphere systems, we may introduce the concept of a propositional selection function.

Definition

A *propositional selection function* is a function $f_{\mathscr{S}} : \mathscr{E}_{\mathcal{L}} \to \mathscr{P}(\mathcal{W}_{\mathcal{L}})$ such that $f(S) \subseteq S$ for every $S \in \mathscr{E}_{\mathcal{L}}$.

For spheres systems:

Definition

Let $\mathscr{S}$ be a system of spheres centered on C. Define a propositional selection function $f_{\mathscr{S}} : \mathscr{E}_{\mathcal{L}} \to \mathscr{P}(\mathcal{W}_{\mathcal{L}})$ by setting for every formula ϕ:

$$f_{\mathscr{S}}([\![\phi]\!]) \quad := \quad \min_{\subseteq}(\mathscr{C}_{\phi}) \cap [\![\phi]\!]$$

We call $f_{\mathscr{S}}$ the *Grovean selection function* for $\mathscr{S}$.

Sphere-Based Revision

Definition

Let K be a belief set. A function $*$ is a *sphere-based revision* for K if there is system of spheres $\mathscr{S}$ centered on $[\![K]\!]$ such that for all formulae ϕ:

$$K * \phi = \mathrm{Th}(f_{\mathscr{S}}([\![\phi]\!])).$$

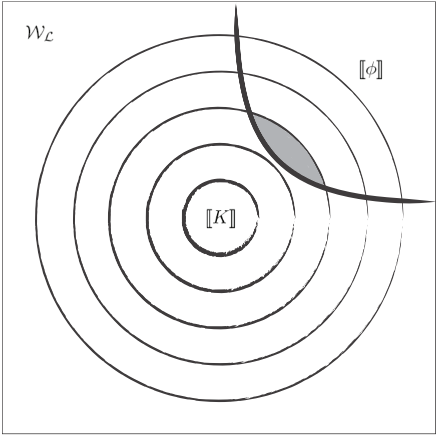
Grove (1988) establishes an important and useful connection between sphere-based revision and the AGM revision postulates.

Theorem (Grove 1988)

Let K be a belief set. Then:

(i) *Every sphere-based revision for K satisfies postulates* $(K * 1)$ *to* $(K * 8)$.

(ii) *Every function on* $\mathrm{For}(\mathcal{L})$ *satisfying* $(K * 1)$ *to* $(K * 8)$ *is a sphere-based revision.*



The third sphere from the center is the least sphere $\min_{\subseteq}(\mathscr{C}_\phi)$ intersecting $[\![\phi]\!]$, and the gray region is the area of the intersection of $\min_{\subseteq}(\mathscr{C}_\phi)$ and $[\![\phi]\!]$, representing the resulting belief state $f_{\mathscr{S}}(\phi)$. The corresponding syntactical representation of $f_{\mathscr{S}}(\phi)$ is given by $K * \phi = \mathrm{Th}(f_{\mathscr{S}}(\phi))$.

The Grove Connection

In fact, (Grove 1988) reveals a close connection between the AGM modeling and the sphere modeling of belief change.

To see this, suppose that $\phi \in K \backslash \mathrm{Cn}(\emptyset)$. To define belief contraction and so belief revision, (AGM 1985) consider the ϕ-remainder set $K \perp \phi$ of maximal subsets Γ of K such that Γ does not imply ϕ.

It is easily verified that there is a one-to-one correspondence $g_\phi : [\![\neg\phi]\!] \to K \perp \phi$ given by $g_\phi(w) = K \cap w$.

Put $K \perp (K \backslash \mathrm{Cn}(\emptyset)) := \bigcup_{\phi \in K \backslash \mathrm{Cn}(\emptyset)} K \perp \phi$ and observe that $\mathcal{W}_\mathcal{L} \backslash [\![K]\!] = \bigcup_{\phi \in K \backslash \mathrm{Cn}(\emptyset)} [\![\neg\phi]\!]$.

Then the family $(g_\phi)_{\phi \in K \backslash \mathrm{Cn}(\emptyset)}$ induces a one-to-one correspondence $G_K : (\mathcal{W}_\mathcal{L} \backslash [\![K]\!]) \to K \perp (K \backslash \mathrm{Cn}(\emptyset))$ given by $G_K(w) := K \cap w$.

The Grove Connection

We arrive at the following result:

Proposition (The Grove Connection, (Grove 1988))

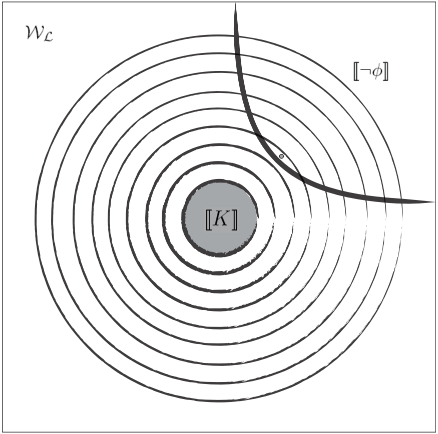
Let K be a belief set. Then there is a bijection $G_K : (\mathcal{W}_{\mathcal{L}} \backslash [\![K]\!]) \rightarrow K \perp (K \backslash \mathrm{Cn}(\emptyset))$ such that for every $\phi \in K \backslash \mathrm{Cn}(\emptyset)$ and $w \in \mathcal{W}_{\mathcal{L}} \backslash [\![K]\!]$:

$$w \in [\![\neg\phi]\!] \quad \text{if and only if} \quad G_K(w) = K \cap w \text{ and } G_K(w) \in K \perp \phi; \tag{1}$$

$$[\![G_K(w)]\!] = [\![K]\!] \cup \{w\}. \tag{2}$$

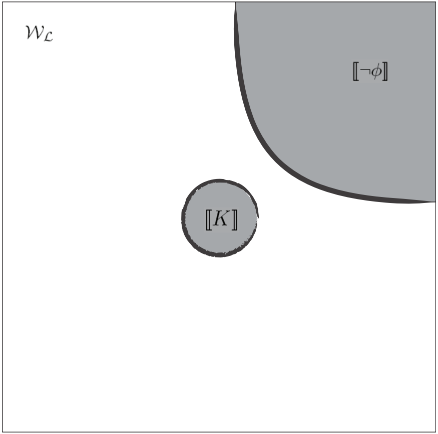
The Grove Connection facilitates the geometric visualization of contraction operators.

Maxichoice Contraction



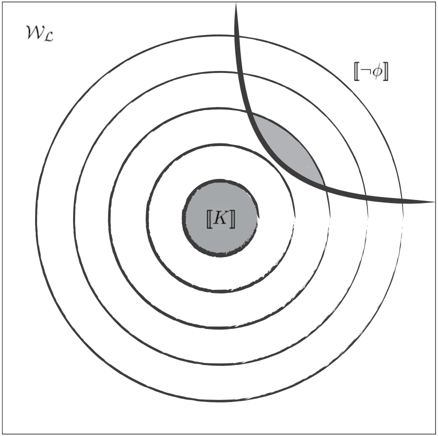
The small gray disc represents the singleton proposition $\{w\}$ selected by $f_{\mathscr{S}}([\![\neg\phi]\!])$, generating the contraction of K by ϕ, $K \dot{-} \phi = K \cap \mathrm{Th}(f_{\mathscr{S}}([\![\neg\phi]\!])) = \mathrm{Th}([\![K]\!] \cup \{w\})$.

Full Meet Contraction



The large gray region in the upper right corner represents the proposition $[\![\neg\phi]\!]$ selected by $f_{\mathscr{S}}([\![\neg\phi]\!])$, generating the contraction of K by ϕ,
$K \dot{-} \phi = K \cap \mathrm{Th}(f_{\mathscr{S}}([\![\neg\phi]\!])) = \mathrm{Th}([\![K]\!] \cup [\![\neg\phi]\!])$.

Partial Meet Contraction



The gray lens represents the proposition given by $f_{\mathscr{S}}([\![\neg\phi]\!])$, generating the contraction of K by ϕ, $K \dot{-} \phi = K \cap \mathrm{Th}(f_{\mathscr{S}}([\![\neg\phi]\!])) = \mathrm{Th}([\![K]\!] \cup f_{\mathscr{S}}([\![\neg\phi]\!]))$.