Logic and Artificial Intelligence

Lecture 19

Eric Pacuit

Currently Visiting the Center for Formal Epistemology, CMU

Center for Logic and Philosophy of Science
Tilburg University

ai.stanford.edu/∼epacuit
e.j.pacuit@uvt.nl

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C. Cross. *'Can' and the Logic of Ability*. Philosophical Studies, Vol. 50, pp. 53 - 64, 1986.

Ability: Reproducibility vs. Reliability

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"Even if opportunity only knocks once, I may be able to act on it, an may be culpable for doing so, or for failing to do so." (p. 1)

M. Brown. *On the Logic of Ability*. Journal of Philosophical Logic, Vol. 17, pp. 1 - 26, 1988.

On the Logic of Ability

$Abl_i \top$

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Brief digression: neighborhood models for modal logic

Neighborhood Frames

Let W be a non-empty set of states.

Any map $N : W \to \wp(\wp(W))$ is called a neighborhood function

Definition

A pair $\langle W, N \rangle$ is a called a neighborhood frame if W a non-empty set and N is a neighborhood function.

Some Terminology

Let $\mathcal{F} = \langle W, N \rangle$ be a neighborhood frame.

- $\mathcal{F}$ is closed under intersections if for any collections of sets $\{X_i\}_{i \in I}$ such that for each $i \in I$, $X_i \in N(w)$, then $\cap_{i \in I} X_i \in N(w)$.
- $\mathcal{F}$ is supplemented, or closed under supersets or monotonic provided for each $X \subseteq W$, if $X \in N(w)$ and $X \subseteq Y \subseteq W$, then $Y \in N(w)$.
- $\mathcal{F}$ contains the unit provided $W \in N(w)$
- the set $\cap_{X \in \mathcal{F}} X$ the core of $\mathcal{F}$. $\mathcal{F}$ contains its core provided $\cap_{X \in \mathcal{F}} X \in \mathcal{F}$.
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From Kripke Frames to Neighborhood Frames

Let $R \subseteq W \times W$, define a map $R^{\rightarrow} : W \rightarrow \wp W$:

$$\text{for each } w \in W \text{, let } R^{\rightarrow}(w) = \{v \mid wRv\}$$

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Definition

Given a relation R on a set W and a state $w \in W$. A set $X \subseteq W$ is *R-necessary at w* if $R^{\rightarrow}(w) \subseteq X$.

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Let $\mathcal{N}_w^R$ be the set of sets that are R-necessary at w:

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Lemma

Let R be a relation on W. Then for each $w \in W$, $\mathcal{N}_w^R$ is augmented.

From Kripke Frames to Neighborhood Frames

Properties of R are reflected in $\mathcal{N}_w^R$:

- If R is reflexive, then for each $w \in W$, $w \in \cap\mathcal{N}_w^R$
- If R is transitive then for each $w \in W$, if $X \in \mathcal{N}_w^R$, then $\{v \mid X \in \mathcal{N}_v^R\} \in \mathcal{N}_w^R$.

From Neighborhood Frames to Kripke Frames

Theorem

- *Let* $\langle W, R \rangle$ *be a relational frame. Then there is an equivalent augmented neighborhood frame.*
- *Let* $\langle W, N \rangle$ *be an augmented neighborhood frame. Then there is an equivalent relational frame.*

From Neighborhood Frames to Kripke Frames

for all $X \subseteq W$, $X \in N(w)$ iff $X \in \mathcal{N}_w^R$.

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Proof.
For each $w \in W$, let $N(w) = \mathcal{N}_w^R$. □

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Theorem

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Proof.

For each $w, v \in W$, $wR_N v$ iff $v \in \cap N(w)$. □

Neighborhood Model

Let $\mathfrak{F} = \langle W, N \rangle$ be a neighborhood frame. A neighborhood model based on $\mathfrak{F}$ is a tuple $\langle W, N, V \rangle$ where $V : \mathsf{At} \to 2^W$ is a valuation function.

Truth in a Model

- $\mathfrak{M}, w \models p$ iff $w \in V(p)$
- $\mathfrak{M}, w \models \neg\varphi$ iff $\mathfrak{M}, w \not\models \varphi$
- $\mathfrak{M}, w \models \varphi \land \psi$ iff $\mathfrak{M}, w \models \varphi$ and $\mathfrak{M}, w \models \psi$

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- $\mathfrak{M}, w \models \varphi \wedge \psi$ iff $\mathfrak{M}, w \models \varphi$ and $\mathfrak{M}, w \models \psi$
- $\mathfrak{M}, w \models \Box\varphi$ iff $(\varphi)^{\mathfrak{M}} \in N(w)$
- $\mathfrak{M}, w \models \Diamond\varphi$ iff $W - (\varphi)^{\mathfrak{M}} \notin N(w)$

where $(\varphi)^{\mathfrak{M}} = \{w \mid \mathfrak{M}, w \models \varphi\}$.

Detailed Example

Suppose $W = \{w, s, v\}$ is the set of states and define a neighborhood model $\mathfrak{M} = \langle W, N, V \rangle$ as follows:

- $N(w) = \{\{s\}, \{v\}, \{w, v\}\}$
- $N(s) = \{\{w, v\}, \{w\}, \{w, s\}\}$
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Further suppose that $V(p) = \{w, s\}$ and $V(q) = \{s, v\}$.

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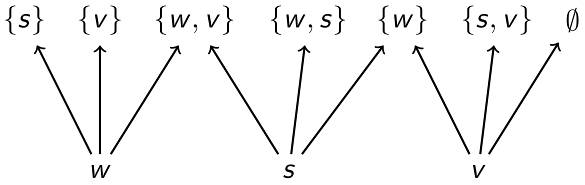

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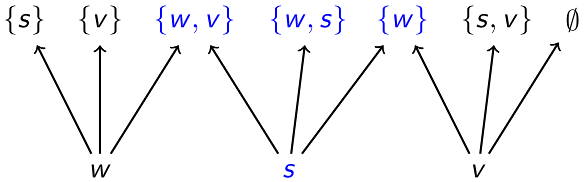

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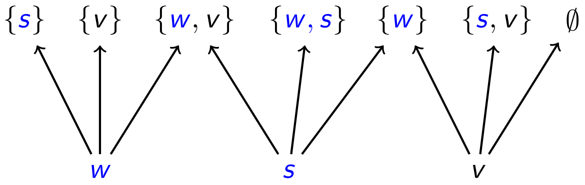

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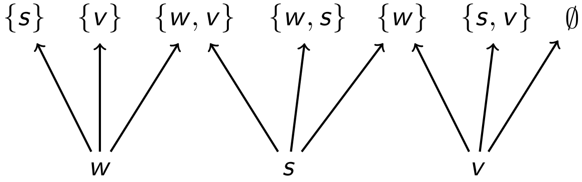

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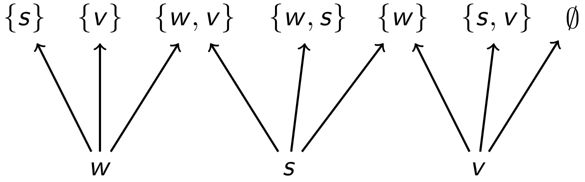

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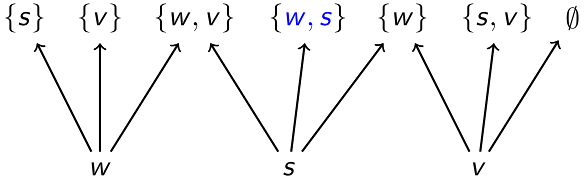

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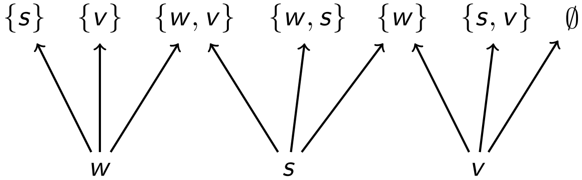

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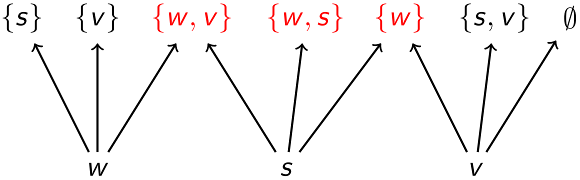

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$$(\neg p)^{\mathfrak{M}} = \{v\}$$

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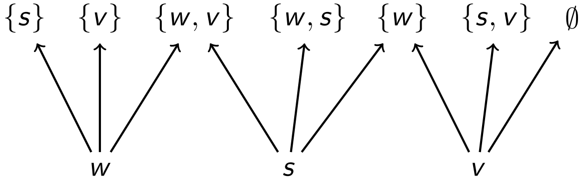

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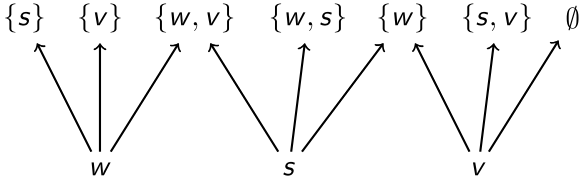

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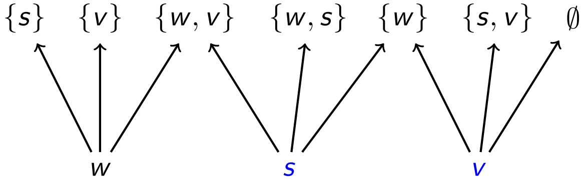

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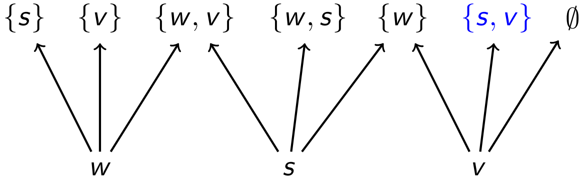

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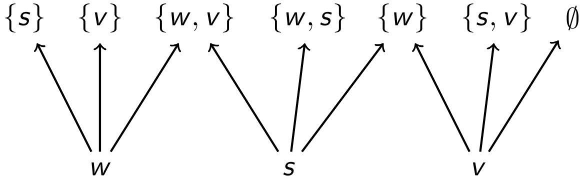

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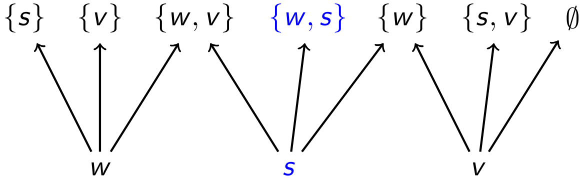

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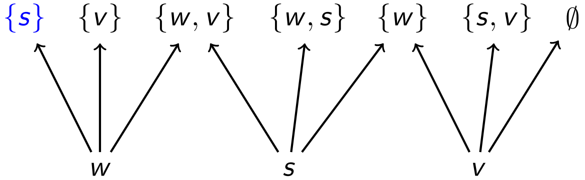

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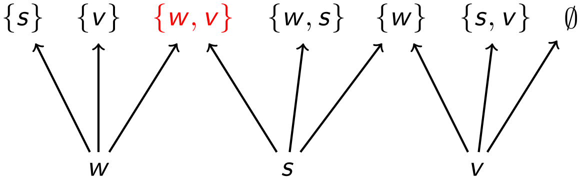

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Other modal operators

- $\mathfrak{M}, w \models \langle\ \rangle\varphi$ iff $\exists X \in N(w)$ such that $\exists v \in X$, $\mathfrak{M}, v \models \varphi$
- $\mathfrak{M}, w \models [\ ]\varphi$ iff $\forall X \in N(w)$ such that $\forall v \in X$, $\mathfrak{M}, v \models \varphi$

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Lemma

Let $\mathfrak{M} = \langle W, N, V\rangle$ be a neighborhood model. The for each $w \in W$,

1. *if $\mathfrak{M}, w \models \Box\varphi$ then $\mathfrak{M}, w \models \langle\ ]\varphi$*
2. *if $\mathfrak{M}, w \models [\ \rangle\varphi$ then $\mathfrak{M}, w \models \Diamond\varphi$*

However, the converses of the above statements are false.

Other modal operators

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Lemma

1. *If $\varphi \to \psi$ is valid in $\mathfrak{M}$, then so is $\langle\ ]\varphi \to \langle\ ]\psi$.*
2. $\langle\ ](\varphi \wedge \psi) \to (\langle\ ]\varphi \wedge \langle\ ]\psi)$ *is valid in* $\mathfrak{M}$

Investigate analogous results for the other modal operators defined above.

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A multi-relational Kripke model is a triple $\mathbb{M} = \langle W, \mathcal{R}, V \rangle$ where $\mathcal{R} \subseteq \wp(W \times W)$.

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A world is called queer if nothing is necessary and everything is possible.

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A multi-relational model with queer worlds is a quadruple $\mathbb{M} = \langle W, Q, \mathcal{R}, V \rangle$.

$\mathbb{M}, w \models \Box\varphi$ iff $w \notin Q$ and $\exists R \in \mathcal{R}$ such that $\forall v \in W$, if wRv then $\mathbb{M}, v \models \varphi$.

Different Semantics

M. Fitting. *Proof Methods for Modal and Intuitionistic Logics*. 1983.

L. Goble. *Multiplex semantics for Deontic Logic*. Nordic Journal of Philosophical Logic (2000).

G. Governatori and A. Rotolo. *On the axiomatization of Elgesems logic of agency and ability*. JPL (2005).

D. Elgesem. *The modal logic of agency*. Nordic Journal of Philosophical Logic 2(2), 1 - 46, 1997.

G. Governatori and A. Rotolo. *On the Axiomatisation of Elgesem's Logic of Agency and Ability*. Journal of Philosophical Logic, 34, pgs. 403 - 431 (2005).

A Minimal Logic of Abilities

$C\varphi$ means "the agent is capable of realizing φ"

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$C\varphi$ means "the agent is capable of realizing φ"

$E\varphi$ means "the agent does bring about φ"

1. All propositional tautologies
2. $\neg C\top$
3. $E\varphi \wedge E\psi \to E(\varphi \wedge \psi)$
4. $E\varphi \to \varphi$
5. $E\varphi \to C\varphi$
6. Modus Ponens plus from $\varphi \leftrightarrow \psi$ infer $E\varphi \leftrightarrow E\psi$ and from $\varphi \leftrightarrow \psi$ infer $C\varphi \leftrightarrow C\psi$

Neighborhood Models for Elgesem's Logic

$\mathcal{M} = \langle W, N^C, N^E, V \rangle$ where W a nonempty set of states, $N^C : W \to \wp(\wp(W))$, $N^E : W \to \wp(\wp(W))$ and $V : \mathsf{At} \to \wp(W)$ with

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- $W \notin N^C(w)$
- If $X \in N^E(w)$ and $Y \in N^E(w)$ then $X \cap Y \in N^E(w)$
- If $X \in N^E(w)$ then $w \in X$
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- $\mathcal{M}, w \models E\varphi$ iff $[\![\varphi]\!]_{\mathcal{M}} \in N^E(w)$
- $\mathcal{M}, w \models C\varphi$ iff $[\![\varphi]\!]_{\mathcal{M}} \in N^C(w)$

Theorem Elgesem's axiomatization of capabilities and abilities is sound and complete with respect to the previous class of neighborhood models.

G. Governatori and A. Rotolo. *On the Axiomatisation of Elgesem's Logic of Agency and Ability*. Journal of Philosophical Logic, 34, pgs. 403 - 431 (2005).

Actions

A. Mele (editor). *The Philosophy of Action*. Oxford Readings in Philosophy.

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A. Mele (editor). *The Philosophy of Action*. Oxford Readings in Philosophy.

Many subtle philosophical issues!

Action individuation; primitive actions; action descriptions; intending to A vs. Aing intentionally; etc.

G. Wilson. *Action*. Stanford Encyclopedia of Philosophy.

Actions

1. Actions as *transitions between states, or situations*:

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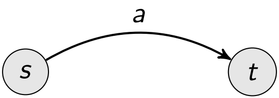

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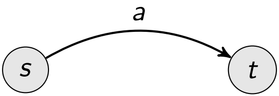
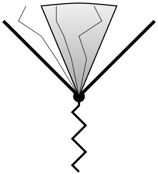
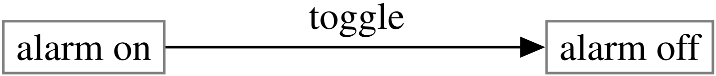


2. Actions *restrict* the set of possible future histories.

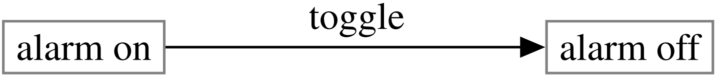


J. van Benthem, H. van Ditmarsch, J. van Eijck and J. Jaspers. *Chapter 6: Propositional Dynamic Logic*. Logic in Action Online Course Project, 2011.

Examples



Examples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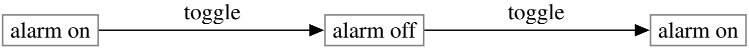
alarm on
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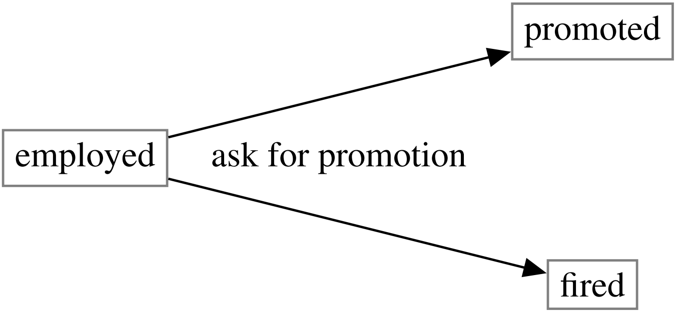
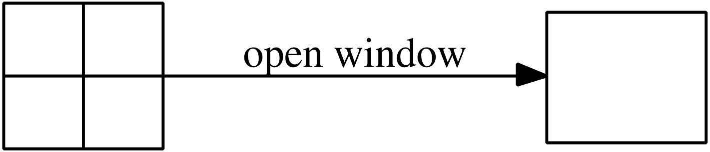
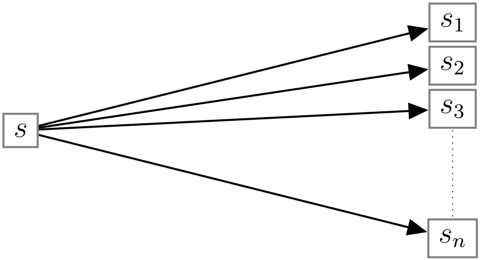
Examples



Examples

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Propositional Dynamic Logic

Language: The language of propositional dynamic logic is generated by the following grammar:

$$p \mid \neg\varphi \mid \varphi \wedge \psi \mid [\alpha]\varphi$$

where $p \in \mathrm{At}$ and α is generated by the following grammar:

$$a \mid \alpha \cup \beta \mid \alpha ; \beta \mid \alpha^* \mid \varphi?$$

where $a \in \mathrm{Act}$ and φ is a formula.

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Semantics: $\mathcal{M} = \langle W, \{R_a \mid a \in \mathrm{P}\}, V\rangle$ where for each $a \in \mathrm{P}$, $R_a \subseteq W \times W$ and $V : \mathrm{At} \to \wp(W)$

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$[\alpha]\varphi$ means "after doing α, φ will be true"

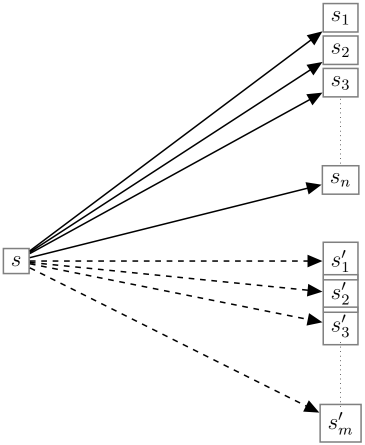
$\langle\alpha\rangle\varphi$ means "after doing α, φ may be true"

$\mathcal{M}, w \models [\alpha]\varphi$ iff for each v, if $wR_\alpha v$ then $\mathcal{M}, v \models \varphi$

$\mathcal{M}, w \models \langle\alpha\rangle\varphi$ iff there is a v such that $wR_\alpha v$ and $\mathcal{M}, v \models \varphi$

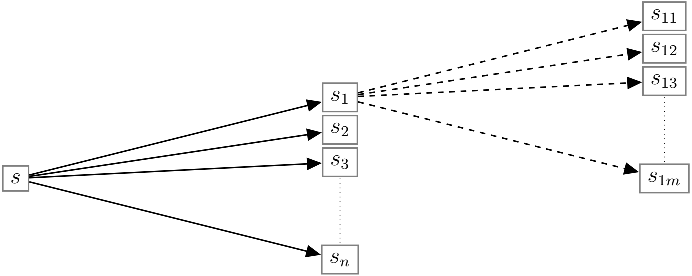
Union

$$R_{\alpha\cup\beta} := R_\alpha \cup R_\beta$$



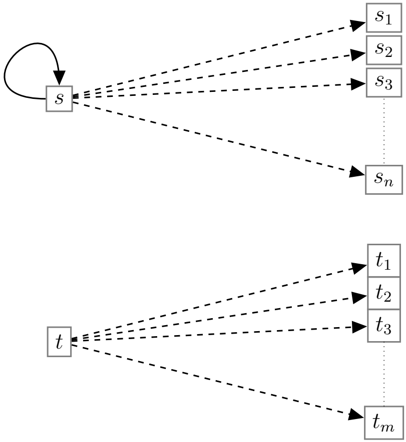
Sequence

$$R_{\alpha;\beta} := R_\alpha \circ R_\beta$$



Test

$$R_{\varphi?} = \{(w, w) \mid \mathcal{M}, w \models \varphi\}$$



Iteration

$$R_{\alpha^*} := \cup_{n \geq 0} R_{\alpha}^{n}$$

Propositional Dynamic Logic

1. Axioms of propositional logic
2. $[\alpha](\varphi \to \psi) \to ([\alpha]\varphi \to [\alpha]\psi)$
3. $[\alpha \cup \beta]\varphi \leftrightarrow [\alpha]\varphi \wedge [\beta]\varphi$
4. $[\alpha; \beta]\varphi \leftrightarrow [\alpha][\beta]\varphi$
5. $[\psi?]\varphi \leftrightarrow (\psi \to \varphi)$
6. $\varphi \wedge [\alpha][\alpha^*]\varphi \leftrightarrow [\alpha^*]\varphi$
7. $\varphi \wedge [\alpha^*](\varphi \to [\alpha]\varphi) \to [\alpha^*]\varphi$
8. Modus Ponens and Necessitation (for each program α)

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4. $[\alpha;\beta]\varphi \leftrightarrow [\alpha][\beta]\varphi$
5. $[\psi?]\varphi \leftrightarrow (\psi \to \varphi)$
6. $\varphi \wedge [\alpha][\alpha^*]\varphi \leftrightarrow [\alpha^*]\varphi$ (Fixed-Point Axiom)
7. $\varphi \wedge [\alpha^*](\varphi \to [\alpha]\varphi) \to [\alpha^*]\varphi$ (Induction Axiom)
8. Modus Ponens and Necessitation (for each program α)

Propositional Dynamic Logic

Theorem PDL is sound and weakly complete with respect to the Segerberg Axioms.

Theorem The satisfiability problem for **PDL** is decidable (EXPTIME-Complete).

D. Kozen and R. Parikh. *A Completeness Proof for Propositional Dynamic Logic*. Theoretical Computer Science, 14, pp. 113-118, 1981.

D. Harel, D. Kozen and Tiuryn. *Dynamic Logic*. MIT Press, 2001.