Logic and Artificial Intelligence

Lecture 24

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Merging Logics of Rational Agency

- ✓ Entangling Knowledge/Beliefs and Preferences
- ✓ "Epistemizing" Logics of Action and Ability
- ✓ BDI (Belief + Desires + Intentions) Logics

Logic and Game Theory

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Logic and Game Theory

Game theory is full of deep puzzles, and there is often disagreement about proposed solutions to them. The puzzlement and disagreement are neither empirical nor mathematical but, rather, concern the meanings of fundamental concepts ('solution', 'rational', 'complete information') and the soundness of certain arguments...Logic appears to be an appropriate tool for game theory both because these conceptual obscurities involve notions such as reasoning, knowledge and counter-factuality which are part of the stock-in-trade of logic, and because it is a prime function of logic to establish the validity or invalidity of disputed arguments.

M.O.L. Bacharach. *Logic and the Epistemic Foundations of Game Theory*. 2001.

(Modal) Logic in Games

M. Pauly and W. van der Hoek. *Modal Logic for Games and Information*. Handbook of Modal Logic (2006).

G. Bonanno. *Modal Logic and Game Theory: Two Alternative Approaches*. Risk Decision and Policy **7** (2002).

J. van Benthem. *Extensive Games as Process Models*. Journal of Logic, Language and Information **11** (2002).

J. Halpern. *A Computer Scientist Looks at Game Theory*. Games and Economic Behavior **45:1** (2003).

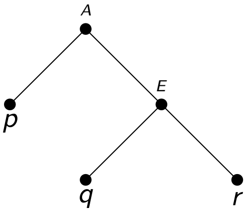
R. Parikh. *Social Software*. Synthese **132: 3** (2002).

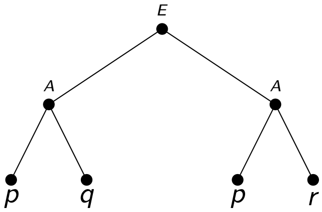
Many topics...

- Social Procedures: Fair-Division Algorithms, Voting Procedures, Cake-Cutting Algorithms
- Logics of rational agency
- Logics of rational interaction
- Game Logics
- When are two games the *same*?
- Epistemic program in game theory
- Social Choice Theory and Logic
- (Formally) Verifying that a social procedure is *correct*
- Develop ("well-behaved") logical languages that can express game theoretic concepts, such as the *Nash equilibrium*

Games for Logic

Games for Logic





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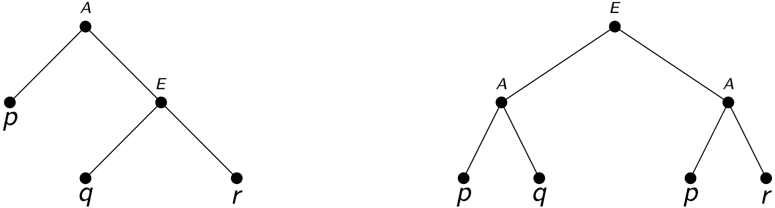
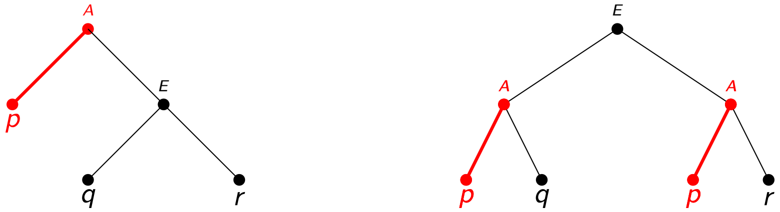


A can force $\{p\}, \{q, r\}$, E can force $\{p, q\}$, $\{p, r\}$

Games for Logic



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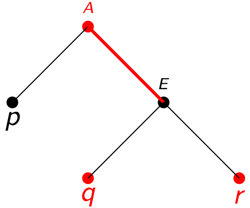

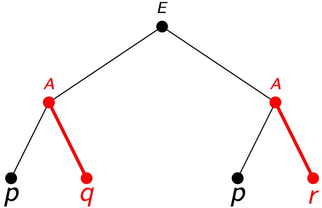

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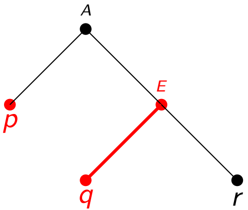

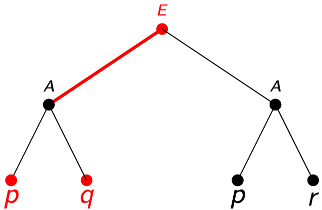

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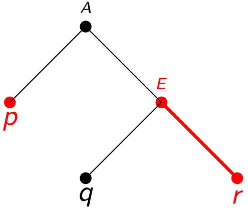

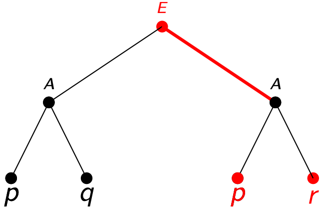

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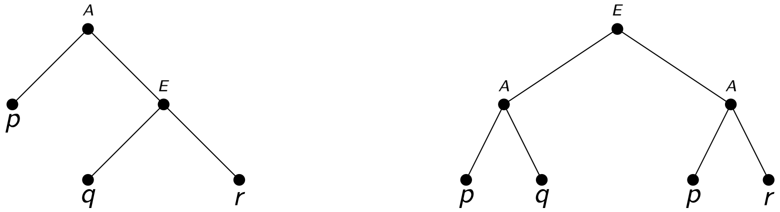

$$p \wedge (q \vee r) \leftrightarrow (p \wedge q) \vee (p \wedge r)$$

A primer on game theoretic models (extensive/normal form games)

Game Situations

1. a group of *self-interested* agents (players) involved in some interdependent decision problem

Game Situations

Bob

L $\quad$ R

1 $\quad$ 0

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Game Situations

	Bob	
	L	R
	1	0
	0	1

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Game Situations

			Bob	
		L		R
Ann	U	1 1		0 0
	D	0 0		1 1

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Game Situations

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Ann		L	R
	U	1,1	0,0
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Game Situations

		Bob L	Bob R
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Ann	D	0,0	1,1

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Game Situations

Bob / Ann

	L	R
U	1,1	0,0
D	0,0	1,1

1. a group of *self-interested* agents (players) involved in some interdependent decision problem, and
2. the players *recognize that they are engaged in a game situation*

Game Situations

		Bob L	Bob R
Ann	U	1,1	0,0
Ann	D	0,0	1,1

What should Ann (Bob) *do*?

Game Situations

Bob

Ann		L	R
	U	1,1	0,0
	D	0,0	1,1

What should Ann (Bob) *do*?

It depends on what she *expects* Bob to do, but this depends on what she thinks Bob expects her to do, and so on...

Just Enough Game Theory

Osborne and Rubinstein. *Introduction to Game Theory*. MIT Press .

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A **game** is a description of strategic interaction that includes

- actions the players *can* take
- description of the players' interests (i.e., preferences),
- description of the "structure" of the decision problem

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A **game** is a description of strategic interaction that includes

- actions the players *can* take
- description of the players' interests (i.e., preferences),
- description of the "structure" of the decision problem

It does not specify the actions that the players do take.

Solution Concepts

A **solution concept** is a systematic description of the outcomes that may emerge in a family of games.

This is the starting point for most of game theory and includes many variants: Nash equilibrium, backwards inductions, or iterated dominance of various kinds.

These are usually thought of as the embodiment of "rational behavior" in some way and used to analyze game situations.

Strategic Games

A **strategic games** is a tuple $\langle N, \{A_i\}_{i \in N}, \{\succeq_i\}_{i \in N} \rangle$ where

- N is a finite set of **players**

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Strategic Games

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- N is a finite set of **players**
- for each $i \in N$, A_i is a nonempty set of **actions**
- for each $i \in N$, $\succeq_i$ is a **preference relation** on $A = \Pi_{i \in N} A_i$ (Often $\succeq_i$ are represented by **utility functions** $u_i : A \to \mathbb{R}$)

Strategic Games: Comments on Preferences

- Preferences may be over a set of consequences C. Assume $g : A \to C$ and $\{\succeq_i^* \mid i \in N\}$ a set of preferences on C. Then for $a, b \in A$,

$$a \succeq_i b \text{ iff } g(a) \succeq_i^* g(b)$$

- Consequences may be affected by exogenous random variable whose realization is not known before choosing actions. Let Ω be a set of states, then define $g : A \times \Omega \to C$. Where $g(a|\cdot)$ is interpreted as a *lottery*.

- Often $\succeq_i$ are represented by **utility functions** $u_i : A \to \mathbb{R}$

Strategic Games: Example

		Column	
		r	l
Row	u		
	d		

- $N = \{Row, Column\}$
- $A_{Row} = \{u, d\}$, $A_{Column} = \{r, l\}$
- $(u, r) \succeq_{Row} (d, l) \succeq_{Row} (u, l) \sim_{Row} (d, r)$
  $(u, r) \succeq_{Column} (d, l) \succeq_{Column} (u, l) \sim_{Column} (d, r)$

Strategic Games: Example

		Column r	Column l
Row	u	(2,2)	(0,0)
Row	d	(0,0)	(1,1)

- $N = \{Row, Column\}$
- $A_{Row} = \{u, d\}$, $A_{Column} = \{r, l\}$
- $u_{Row} : A_{Row} \times A_{Column} \to \{0, 1, 2\}$,
  $u_{Column} : A_{Row} \times A_{Column} \to \{0, 1, 2\}$ with
  $u_{Row}(u, r) = u_{Column}(u, r) = 2$,
  $u_{Row}(d, l) = u_{Column}(d, l) = 2$,
  and $u_x(u, l) = u_x(d, r) = 0$ for $x \in N$.

Nash Equilibrium

Let $\langle N, \{A_i\}_{i \in N}, \{\succeq_i\}_{i \in N} \rangle$ be a strategic game

For $a_{-i} \in A_{-i}$, let

$$B_i(a_{-i}) = \{a_i \in A_i \mid (a_{-i}, a_i) \succeq_i (a_{-i}, a_i') \; \forall \; a_i' \in A_i\}$$

B_i is the **best-response** function.

Nash Equilibrium

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B_i is the **best-response** function.

$a^* \in A$ is a **Nash equilibrium** iff $a_i^* \in B_i(a_{-i}^*)$ for all $i \in N$.

Strategic Games Example: Bach or Stravinsky?

	b_c	s_c
b_r	2,1	0,0
s_r	0,0	1,2

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	b_c	s_c
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$$N = \{r, c\} \quad A_r = \{b_r, s_r\}, A_c = \{b_c, s_c\}$$

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$B_r(b_c) = \{b_r\}$ $\qquad$ $B_r(s_c) = \{s_r\}$

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$$B_r(b_c) = \{b_r\} \qquad B_r(s_c) = \{s_r\}$$

$$B_c(b_r) = \{b_c\} \qquad B_c(s_r) = \{s_c\}$$

(b_r, b_c) is a Nash Equilibrium (s_r, s_c) is a Nash Equilibrium

Another Example: Pure Coordination

		Bob L	Bob R
Ann	U	1,1	0,0
Ann	D	0,0	1,1

Another Example: Hi-Low

		Bob	
		L	R
Ann	U	3,3	0,0
	D	0,0	1,1

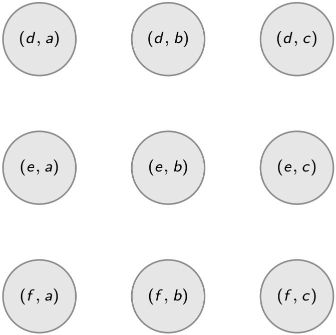
Reasoning *about* (strategic) games

Reasoning *about* (strategic) games

There is Kripke structure "built in" a strategic game.

$W = \{\sigma \mid \sigma \text{ is a strategy profile: } \sigma \in \Pi_{i \in N} S_i\}$

	a	b	c
d	(2,3)	(2,2)	(1,1)
e	(0,2)	(4,0)	(1,0)
f	(0,1)	(1,4)	(2,0)



Reasoning *about* (strategic) games

$\sigma \sim_i \sigma'$ iff $\sigma_i = \sigma'_i$: this epistemic relation represents player i's "view of the game" at the *ex interim* stage where i's choice is fixed but the choices of the other players' are unknown

	a	b	c
d	(2,3)	(2,2)	(1,1)
e	(0,2)	(4,0)	(1,0)
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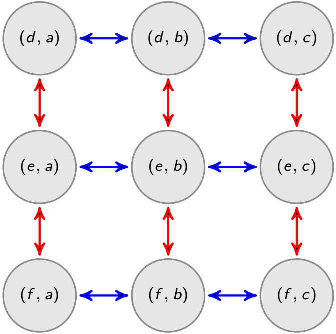
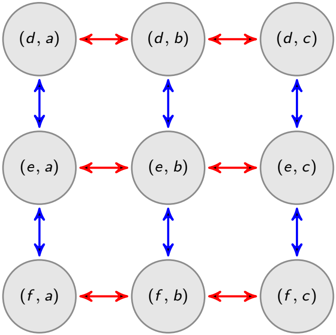


Reasoning *about* (strategic) games

$\sigma \approx_i \sigma'$ iff $\sigma_{-i} = \sigma_{-i}$: this relation of "action freedom" gives the alternative choices for player i when the other players' choices are fixed.

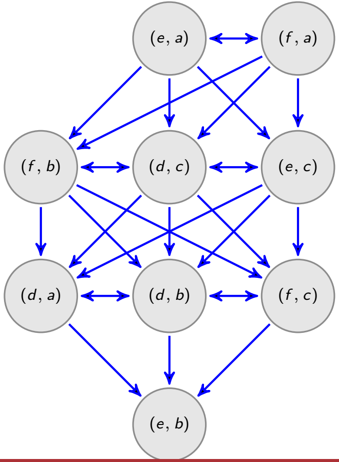
	a	b	c
d	(2,3)	(2,2)	(1,1)
e	(0,2)	(4,0)	(1,0)
f	(0,1)	(1,4)	(2,0)



Reasoning *about* (strategic) games

$\sigma \succeq_i \sigma'$ iff player i prefers the outcome σ at least as much as outcome σ'

	a	b	c
d	(2,3)	(2,2)	(1,1)
e	(0,2)	(4,0)	(1,0)
f	(0,1)	(1,4)	(2,0)



Reasoning about strategic games

$$\mathcal{M} = \langle W, \{\sim_i\}_{i \in N}, \{\approx_i\}_{i \in N}, \{\succeq_i\}_{i \in N} \rangle$$

- $\sigma \models [\sim_i]\varphi$ iff for all σ', if $\sigma \sim_i \sigma'$ then $\sigma' \models \varphi$.
- $\sigma \models [\approx_i]\varphi$ iff for all σ', if $\sigma \approx_i \sigma'$ then $\sigma' \models \varphi$.
- $\sigma \models \langle \succeq_i \rangle\varphi$ iff there exists σ' such that $\sigma' \succeq_i \sigma$ and $\sigma' \models \varphi$.
- $\sigma \models \langle \succ_i \rangle\varphi$ iff there is a σ' with $\sigma' \succeq_i \sigma$, $\sigma \not\succeq_i \sigma'$, and $\sigma' \models \varphi$

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What is the complete logic of *finite* games?

Reasoning about strategic games

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the equivalence $[\sim_i][\approx_i]\varphi \leftrightarrow [\approx_i][\sim_i]\varphi$ is valid on *full* games

Reasoning about strategic games

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Can we modally define the Nash Equilibrium?

Reasoning about strategic games

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Can we modally define the Nash Equilibrium? $Nash \ := \ \bigwedge_{i \in N} Br_i$

Reasoning about strategic games

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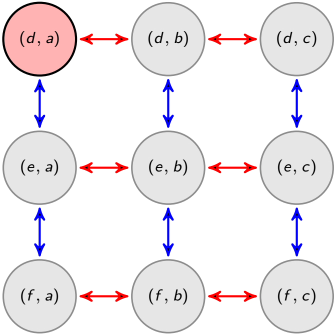
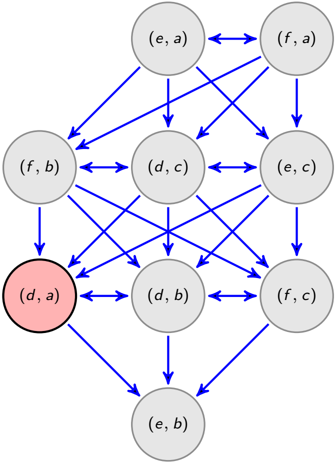
Can we modally define the *best response* for i?

Reasoning about strategic games

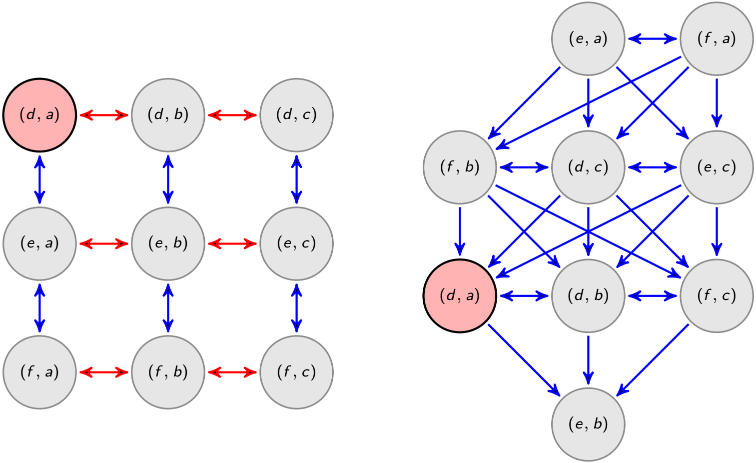
	a	b	c
d	(2,3)	(2,2)	(1,1)
e	(0,2)	(4,0)	(1,0)
f	(0,1)	(1,4)	(2,0)

Reasoning about strategic games



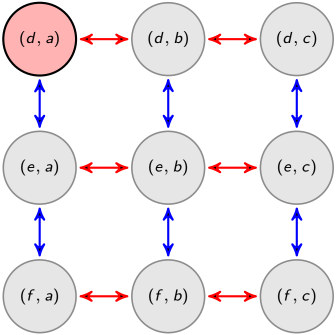


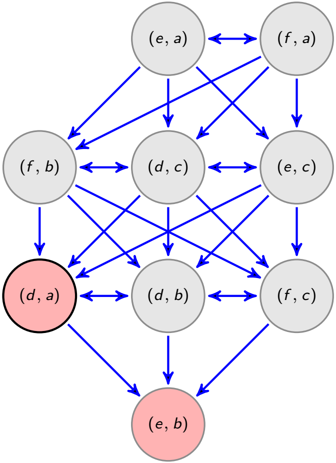
Reasoning about strategic games



$\sigma \models \neg\langle\succeq_i\rangle\top$: there is no outcome at least as good as σ

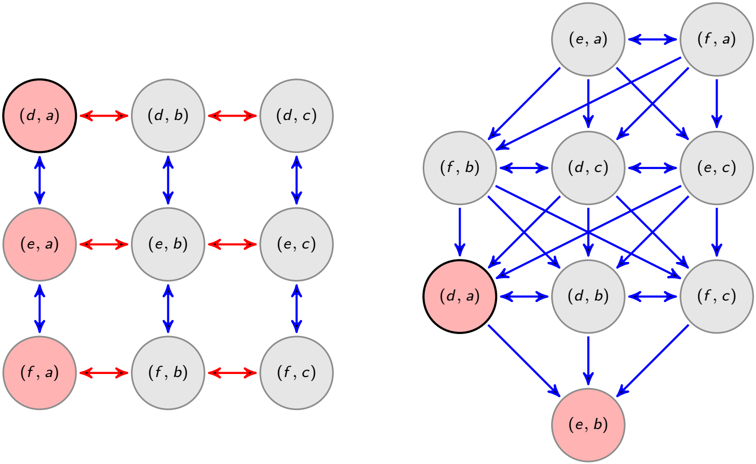
Reasoning about strategic games





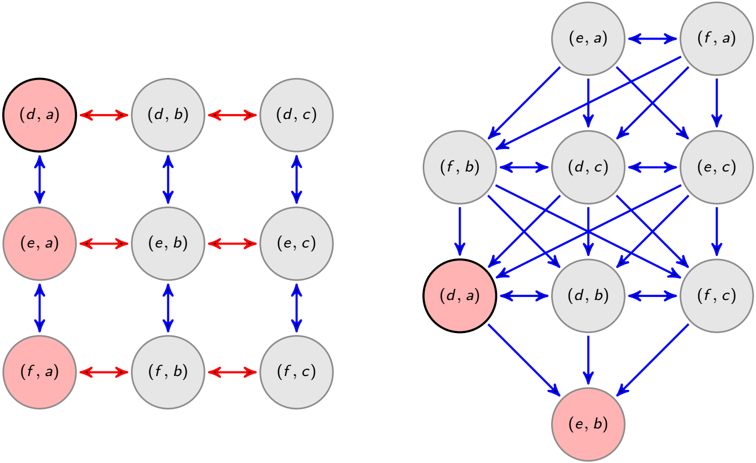
$(d, a) \not\models \neg\langle \succeq_i \rangle \top$: there is no outcome at least as good as σ

Reasoning about strategic games



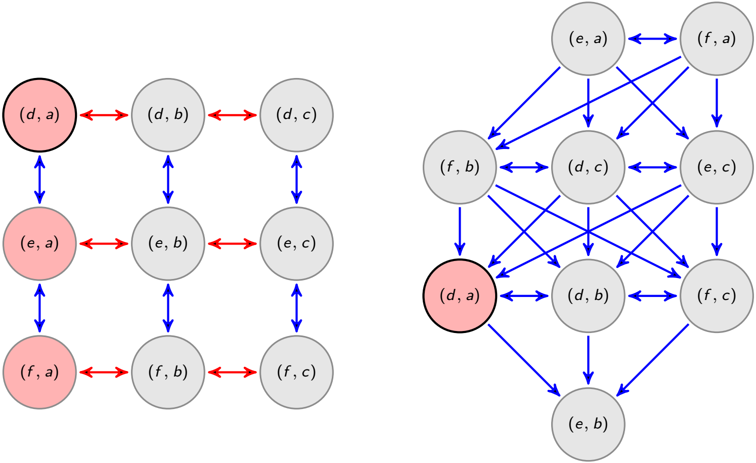
there is no outcome *which i can choose* that is at least as good

Reasoning about strategic games



$\sigma \models \langle \approx_i \cap \succ_i \rangle \varphi$ iff there is σ' such that $\sigma(\approx_i \cap \succ_i)\sigma'$ and $\sigma' \models \varphi$

Reasoning about strategic games



the best response for player i is defined as $\neg\langle \approx_i \cap \succ_i \rangle \top$

Reasoning about extensive games

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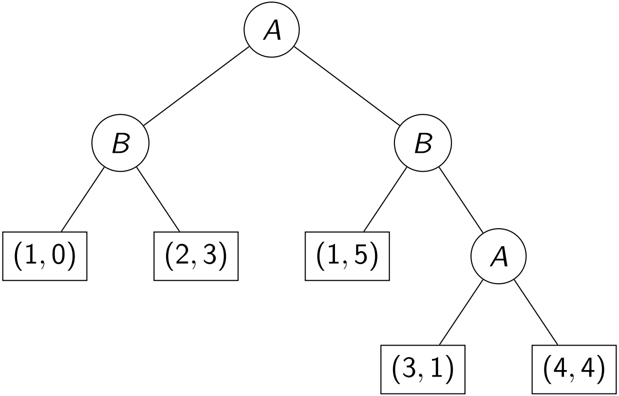
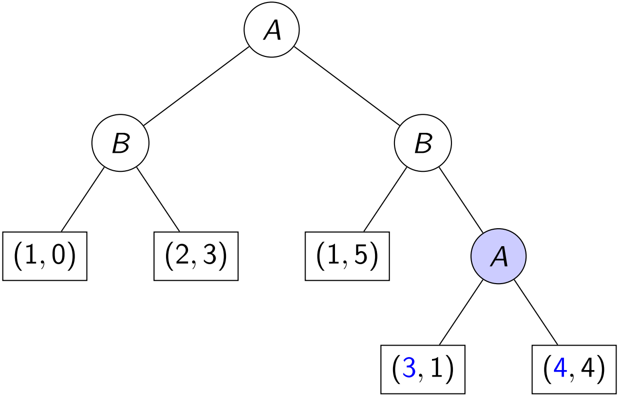


Reasoning about extensive games

Invented by Zermelo, Backwards Induction is an iterative algorithm for "solving" and extensive game.

Reasoning about extensive games



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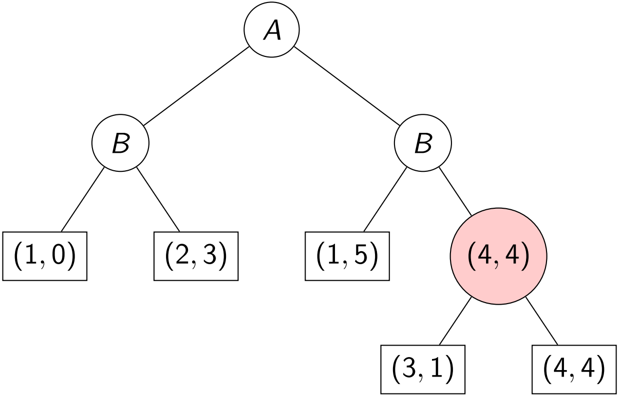

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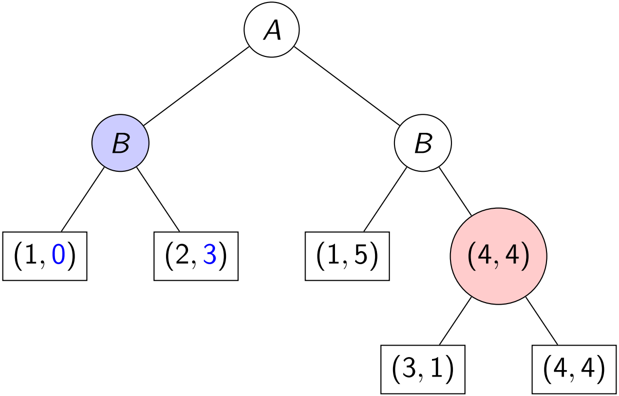

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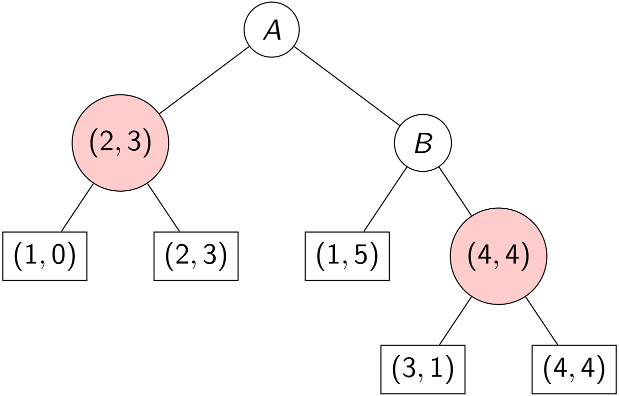

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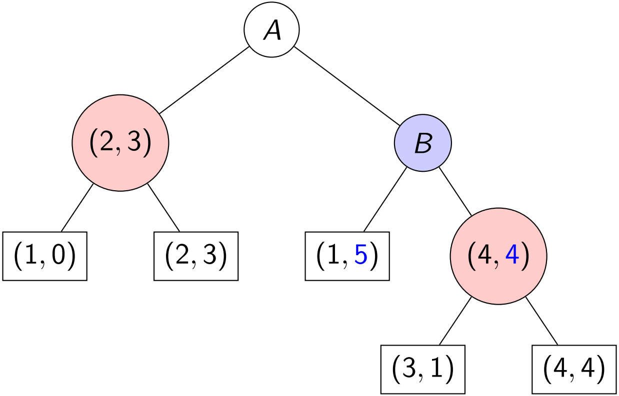

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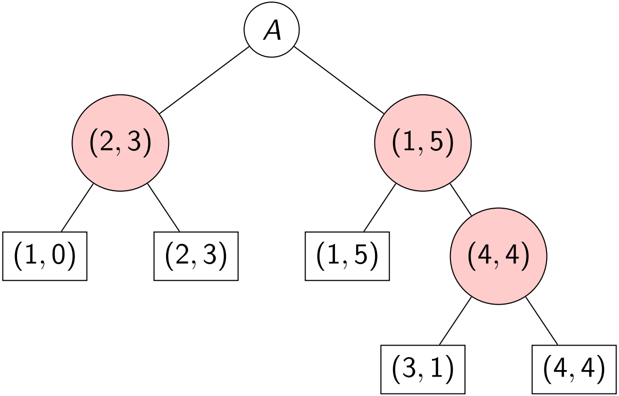

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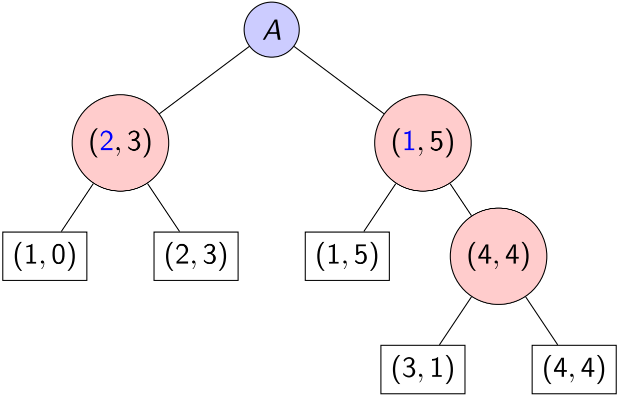

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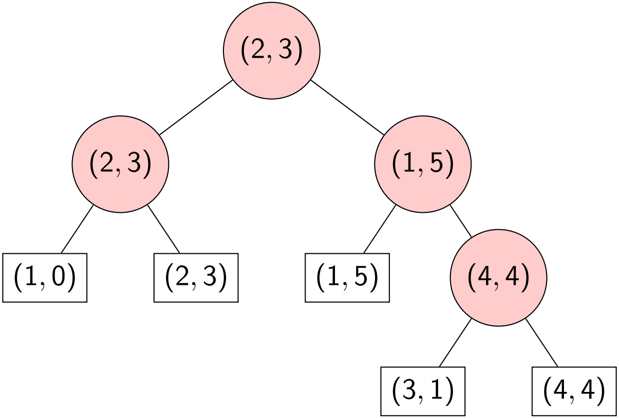

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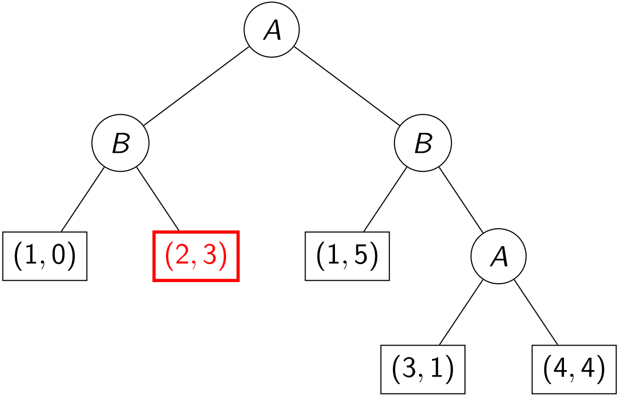

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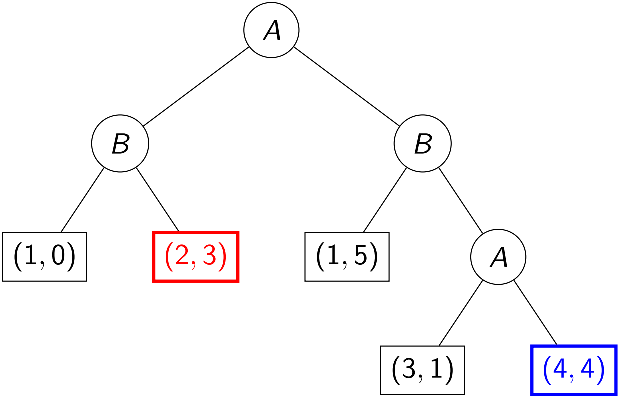

Characterizing Backwards Induction

For each extensive game form, the strategy profile σ is a backward induction solution iff σ is played at the root of a tree satisfying the following modal axiom for all propositions p and players i:

$$(turn_i \wedge \langle \sigma^* \rangle (end \wedge p)) \rightarrow [move_i] \langle \sigma^* \rangle (end \wedge \langle \succeq_i \rangle p)$$

$move_i = \bigcup_{a \text{ is an } i\text{-move}} a$, $turn_i$ is a propositional variable saying that it is i's turn to move, and *end* is a propositional variable true at only end nodes

J. van Benthem, S. van Otterloo and O. Roy. *Preference Logic, Conditionals, and Solution Concepts in Games*. In Modality Matters: Twenty-Five Essays in Honour of Krister Segerberg, 2006.

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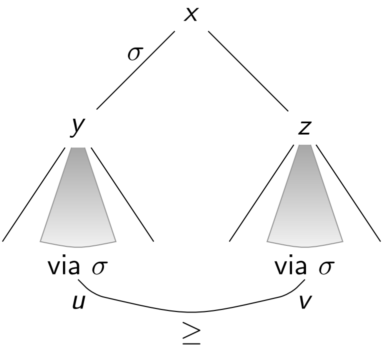

Modal Languages for Games

- $[a \cup b]\langle c \cup d \rangle p$: "for each choice between *a* or *b* there is a choice between *c* or *d* ending in a *p*-state.

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  $K_A\Box_E\varphi$ vs. $\Box_E K_A\varphi$
- preferences, ...

Reasoning *with* games

Background: Propositional Dynamic Logic

Let P be a set of atomic programs and At a set of atomic propositions.

Formulas of **PDL** have the following syntactic form:

$$\varphi := p \mid \bot \mid \neg\varphi \mid \varphi \vee \psi \mid [\alpha]\varphi$$

$$\alpha := a \mid \alpha \cup \beta \mid \alpha ; \beta \mid \alpha^* \mid \varphi ?$$

where $p \in$ At and $a \in$ P.

$[\alpha]\varphi$ is intended to mean "after executing the program α, φ is true"

Background: Propositional Dynamic Logic

Semantics: $\mathcal{M} = \langle W, \{R_a \mid a \in \mathsf{P}\}, V \rangle$ where for each $a \in \mathsf{P}$, $R_a \subseteq W \times W$ and $V : \mathsf{At} \to \wp(W)$

- $R_{\alpha \cup \beta} := R_\alpha \cup R_\beta$
- $R_{\alpha;\beta} := R_\alpha \circ R_\beta$
- $R_{\alpha^*} := \cup_{n \geq 0} R_\alpha^n$
- $R_{\varphi?} = \{(w, w) \mid \mathcal{M}, w \models \varphi\}$

$$\mathcal{M}, w \models [\alpha]\varphi \text{ iff for each } v, \text{ if } wR_\alpha v \text{ then } \mathcal{M}, v \models \varphi$$

Background: Propositional Dynamic Logic

1. Axioms of propositional logic
2. $[\alpha](\varphi \to \psi) \to ([\alpha]\varphi \to [\alpha]\psi)$
3. $[\alpha \cup \beta]\varphi \leftrightarrow [\alpha]\varphi \wedge [\beta]\varphi$
4. $[\alpha; \beta]\varphi \leftrightarrow [\alpha][\beta]\varphi$
5. $[\psi?]\varphi \leftrightarrow (\psi \to \varphi)$
6. $\varphi \wedge [\alpha][\alpha^*]\varphi \leftrightarrow [\alpha^*]\varphi$
7. $\varphi \wedge [\alpha^*](\varphi \to [\alpha]\varphi) \to [\alpha^*]\varphi$
8. Modus Ponens and Necessitation (for each program α)

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6. $\varphi \land [\alpha][\alpha^*]\varphi \leftrightarrow [\alpha^*]\varphi$ (Fixed-Point Axiom)
7. $\varphi \land [\alpha^*](\varphi \to [\alpha]\varphi) \to [\alpha^*]\varphi$ (Induction Axiom)
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From **PDL** to Game Logic

R. Parikh. *The Logic of Games and its Applications.*. Annals of Discrete Mathematics. (1985) .

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Main Idea:

In **PDL**: $w \models \langle\pi\rangle\varphi$: there is a run of the program π starting in state w that ends in a state where φ is true.

The programs in **PDL** can be thought of as *single player games*.

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Game Logic generalized **PDL** by considering two players:

In **GL**: $w \models \langle\gamma\rangle\varphi$: Angel has a **strategy** in the game γ to ensure that the game ends in a state where φ is true.

From **PDL** to Game Logic

Consequences of two players:

From **PDL** to Game Logic

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$\langle\gamma\rangle\varphi$: Angel has a strategy in γ to ensure φ is true

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However, $[\gamma]\varphi \wedge [\gamma]\psi \rightarrow [\gamma](\varphi \wedge \psi)$ is **not** a valid principle

From **PDL** to Game Logic

Reinterpret operations and invent new ones:

- $?\varphi$: Check whether φ currently holds
- $\gamma_1;\gamma_2$: First play γ_1 then γ_2
- $\gamma_1 \cup \gamma_2$: Angel choose between γ_1 and γ_2
- γ^*: Angel can choose how often to play γ (possibly not at all); each time she has played γ, she can decide whether to play it again or not.
- γ^d: Switch roles, then play γ
- $\gamma_1 \cap \gamma_2 := (\gamma_1^d \cup \gamma_2^d)^d$: Demon chooses between γ_1 and γ_2
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Game Logic

Syntax

Let Γ_0 be a set of atomic games and At a set of atomic propositions. Then formulas of Game Logic are defined inductively as follows:

$$\gamma \ := \ g \mid \varphi? \mid \gamma;\gamma \mid \gamma \cup \gamma \mid \gamma^* \mid \gamma^d$$
$$\varphi \ := \ \bot \mid p \mid \neg\varphi \mid \varphi \vee \varphi \mid \langle\gamma\rangle\varphi \mid [\gamma]\varphi$$

where $p \in \mathrm{At}, g \in \Gamma_0$.

Game Logic

A neighborhood game model is a tuple
$\mathcal{M} = \langle W, \{E_g \mid g \in \Gamma_0\}, V\rangle$ where

W is a nonempty set of states

For each $g \in \Gamma_0$, $E_g : W \to \wp(\wp(W))$ is a monotonic neighborhood function.

$X \in E_g(w)$ means in state s, Angel has a strategy to force the game to end in *some* state in X (we may write wE_gX)

$V : \mathrm{At} \to \wp(W)$ is a valuation function.

Game Logic

Propositional letters and boolean connectives are as usual.

$\mathcal{M}, w \models \langle\gamma\rangle\varphi$ iff $(\varphi)^{\mathcal{M}} \in E_\gamma(w)$

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Suppose $E_\gamma(Y) := \{s \mid Y \in E_g(s)\}$

- $E_{\gamma_1;\gamma_2}(Y) \;:=\; E_{\gamma_1}(E_{\gamma_2}(Y))$
- $E_{\gamma_1\cup\gamma_2}(Y) \;:=\; E_{\gamma_1}(Y) \cup E_{\gamma_2}(Y)$
- $E_{\varphi?}(Y) \;:=\; (\varphi)^{\mathcal{M}} \cap Y$
- $E_{\gamma^d}(Y) \;:=\; \overline{E_\gamma(\overline{Y})}$
- $E_{\gamma^*}(Y) \;:=\; \mu X . Y \cup E_\gamma(X)$

Game Logic: Axioms

1. All propositional tautologies
2. $\langle\alpha;\beta\rangle\varphi \leftrightarrow \langle\alpha\rangle\langle\beta\rangle\varphi$ Composition
3. $\langle\alpha\cup\beta\rangle\varphi \leftrightarrow \langle\alpha\rangle\varphi \vee \langle\beta\rangle\varphi$ Union
4. $\langle\psi?\rangle\varphi \leftrightarrow (\psi \wedge \varphi)$ Test
5. $\langle\alpha^{d}\rangle\varphi \leftrightarrow \neg\langle\alpha\rangle\neg\varphi$ Dual
6. $(\varphi \vee \langle\alpha\rangle\langle\alpha^*\rangle\varphi) \rightarrow \langle\alpha^*\rangle\varphi$ Mix

and the rules,

$$\frac{\varphi \qquad \varphi \rightarrow \psi}{\psi} \qquad \frac{\varphi \rightarrow \psi}{\langle\alpha\rangle\varphi \rightarrow \langle\alpha\rangle\psi} \qquad \frac{(\varphi \vee \langle\alpha\rangle\psi) \rightarrow \psi}{\langle\alpha^*\rangle\varphi \rightarrow \psi}$$

Game Logic

- Game Logic is more expressive than **PDL**

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$$\langle (g^d)^* \rangle \bot$$

Game Logic

- Game Logic is more expressive than **PDL**

$$\langle (g^d)^* \rangle \bot$$

- All GL games are determined.

Game Logic

Theorem Dual-free game logic is sound and complete with respect to the class of all game models.

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Open Question Is (full) game logic complete with respect to the class of all game models?

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M. Pauly. *Logic for Social Software*. Ph.D. Thesis, University of Amsterdam (2001).