Logic and Artificial Intelligence

Lecture 25

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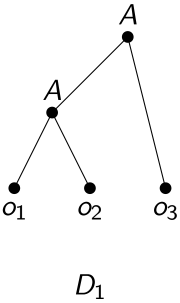
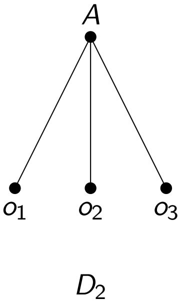
When are two games the *same*?

When are two games the *same*?

- Whose point-of-view? (players, modelers)
- Game-theoretic analysis should not depend on "irrelevant" mathematical details
- Different perspectives: transformations, structural, agent

The same decision problem



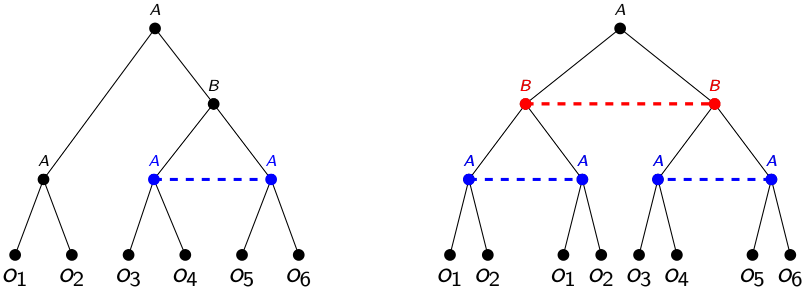


Thompson Transformations

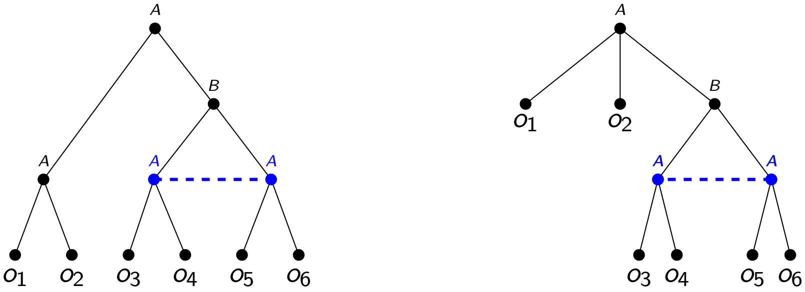
Game-theoretic analysis should not depend on “irrelevant” features of the (mathematical) description of the game.

F. B. Thompson. *Equivalence of Games in Extensive Form*. Classics in Game Theory, pgs 36 - 45, 1952.

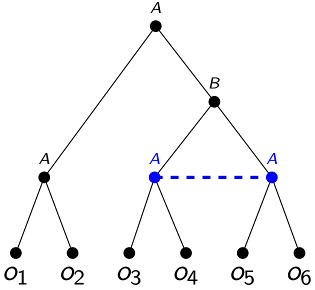
(Osborne and Rubinstein, pgs. 203 - 212)



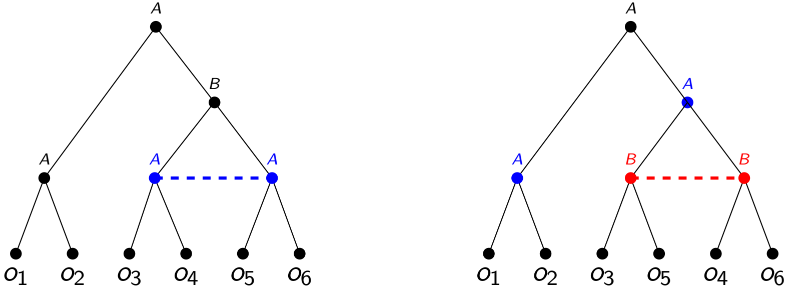
Addition of Superfluous Move



Coalescing of moves



Inflation/deflation



Interchange of moves

Theorem (Thompson) Each of the previous transformations preserves the reduced strategic form of the game. In finite extensive games (without uncertainty between subhistories), if any two games have the same reduced normal form then one can be obtained from the other by a sequence of the four transformations.

Other transformations/game forms

Kohlberg and Mertens. *On Strategic Stability of Equilibria*. Econometrica (1986).

Elmes and Reny. *On The Strategic Equivalence of Extensive Form Games*. Journal of Economic Theory (1994).

G. Bonanno. *Set-Theoretic Equivalence of Extensive-Form Games*. IJGT (1992).

Games as Processes

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- Extensive games are natural process models which support many familiar modal logics such as *bisimulation*.
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J. van Benthem. *Extensive Games as Process Models*. IJGT, 2001.

Game Algebra

Definition Two games γ_1 and γ_2 are equivalent provided $E_{\gamma_1} = E_{\gamma_2}$ in all models

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Definition Two games γ_1 and γ_2 are equivalent provided $E_{\gamma_1} = E_{\gamma_2}$ in all models (iff $\langle\gamma_1\rangle p \leftrightarrow \langle\gamma_2\rangle p$ is valid for a p which occurs neither in γ_1 nor in γ_2.)

Game Algebra

Game Boards: Given a set of states or positions B, for each game g and each player i there is an associated relation $E_g^i \subseteq B \times 2^B$:

$pE_g^i T$ holds if in position p, i can force that the outcome of g will be a position in T.

- (monotonicity) if $pE_g^i T$ and $T \subseteq U$ then $pE_g^i U$
- (consistency) if $pE_g^i T$ then not $pE_g^{1-i}(B - T)$

Given a game board (a set B with relations E_g^i for each game and player), we say that two games g, h ($g \approx h$) are equivalent if $E_g^i = E_h^i$ for each i.

Game Algebra

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4. $-x; -y \approx -(x; y)$
5. $y \preceq z \Rightarrow x; y \preceq x; z$

Theorem Sound and complete axiomatizations of (iteration free) game algebra

Y. Venema. *Representing Game Algebras*. Studia Logica **75** (2003).

V. Goranko. *The Basic Algebra of Game Equivalences*. Studia Logica **75** (2003).

- Reasoning about strategic/extensive games
- Reasoning with strategic/extensive games
- Reasoning in strategic/extensive games

Rationality in Interaction

What does it mean to be rational when the outcome of one's action depends upon the actions of other people and everyone is trying to guess what the others will do?

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In social interaction, rationality has to be enriched with further assumptions about individuals' **mutual knowledge and beliefs**, *but these assumptions are not without consequence.*

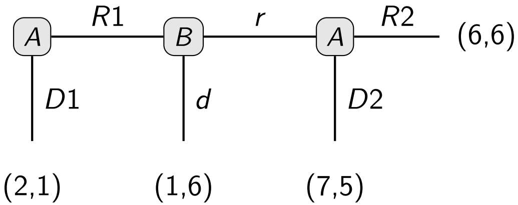
C. Bicchieri. *Rationality and Game Theory*. Chapter 10 in *Handbook of Rationality*.

Key Assumptions

CK1 The structure of the game, including players' strategy sets and payoff functions, is common knowledge among the players.

CK2 The players are rational (i.e., they are expected utility maximizers) and this is common knowledge.

BI Puzzle



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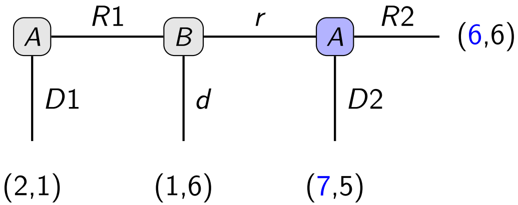

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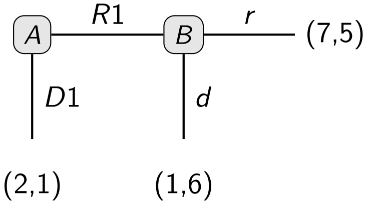

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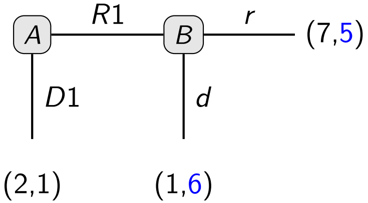

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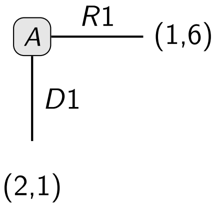

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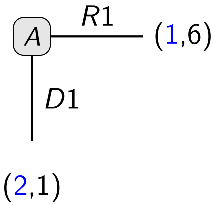

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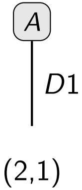

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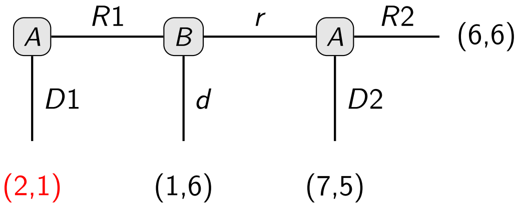
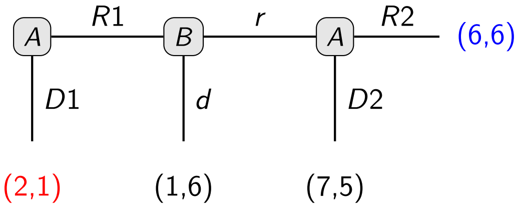


But what if...



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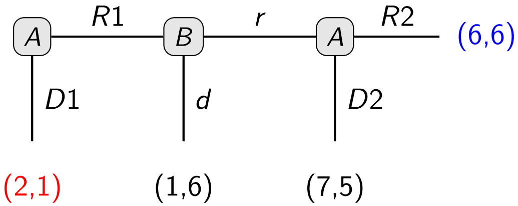

- Are the players *irrational*?
- What argument leads to the BI solution?

R. Aumann. *Backwards induction and common knowledge of rationality*. Games and Economic Behavior, 8, pgs. 6 - 19, 1995.

R. Stalnaker. *Knowledge, belief and counterfactual reasoning in games*. Economics and Philosophy, 12, pgs. 133 - 163, 1996.

J. Halpern. *Substantive Rationality and Backward Induction*. Games and Economic Behavior, 37, pp. 425-435, 1998.

Models of Extensive Games

Let Γ be a *non-degenerate* extensive game with perfect information. Let Γ_i be the set of nodes controlled by player i.

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(A1) If $w \sim_i w'$ then $\sigma_i(w) = \sigma_i(w')$.

Rationality

$h_i^v(\sigma)$ denote "i's payoff if σ is followed from node v"

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i **is rational at** v **in** w provided for all strategies $s_i \neq \sigma_i(w)$, $h_i^v(\sigma(w')) \geq h_i^v((\sigma_{-i}(w'), s_i))$ for some $w' \in [w]_i$.

Substantive Rationality

i is **substantively rational** in state w if i is rational at a vertex v in w of every vertex in $v \in \Gamma_i$

Stalnaker Rationality

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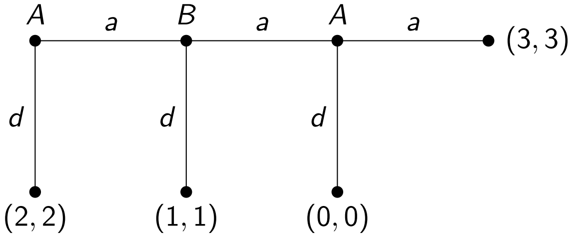
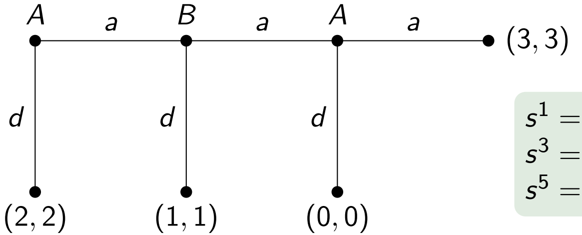
(F1) v is reached in $f(w, v)$ (i.e., v is on the path determined by $\sigma(f(w, v))$)

(F2) If v is reached in w, then $f(w, v) = w$

(F3) $\sigma(f(w, v))$ and $\sigma(w)$ agree on the subtree of Γ below v


A
a
B
a
A
a
(3, 3)
d
d
d
(2, 2)
(1, 1)
(0, 0)



$s^1 = (da, d)$, $s^2 = (aa, d)$,
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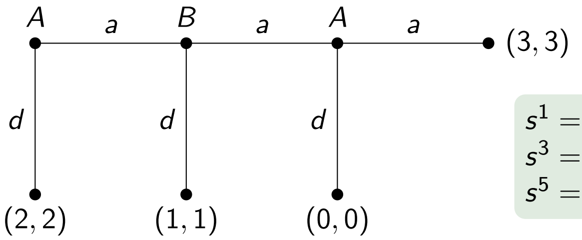

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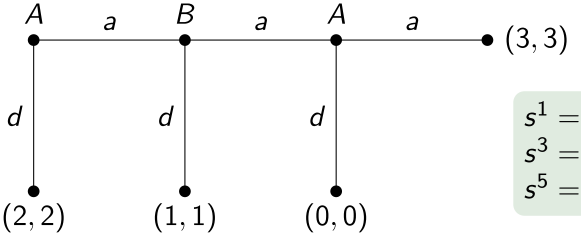
- $W = \{w_1, w_2, w_3, w_4, w_5\}$ with $\sigma(w_i) = s^i$
- $[w_i]_A = \{w_i\}$ for $i = 1, 2, 3, 4, 5$
- $[w_i]_B = \{w_i\}$ for $i = 1, 4, 5$ and $[w_2]_B = [w_3]_B = \{w_2, w_3\}$



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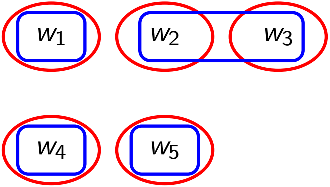

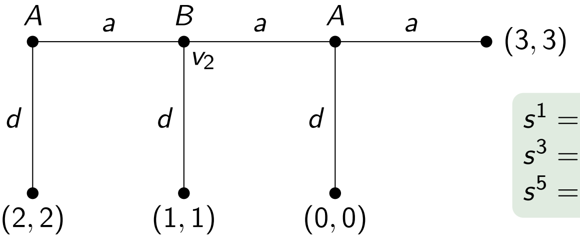

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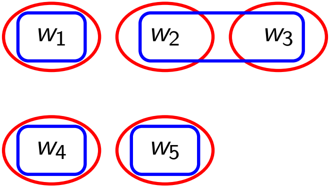
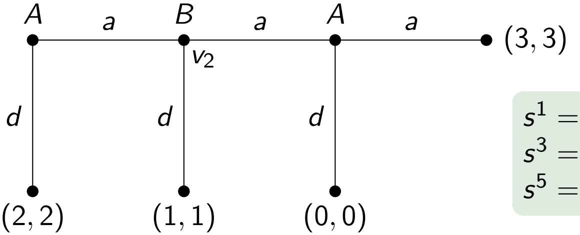


It is **common knowledge** at w_1 that if vertex v_2 were reached, Bob would play down.



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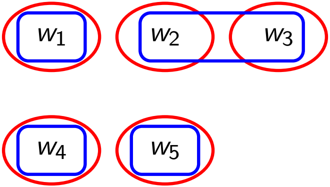
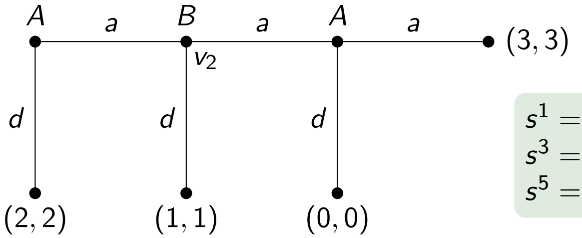


Bob is not rational at v_2 in w_1



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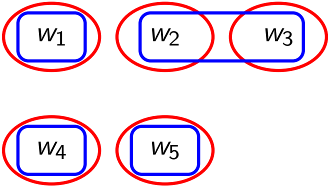
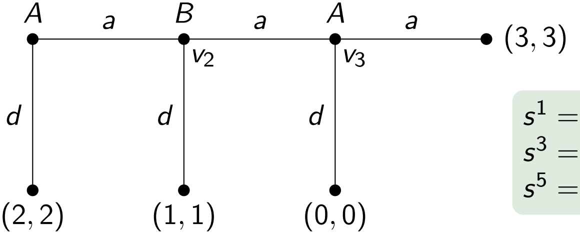


Bob is rational at v_2 in w_2



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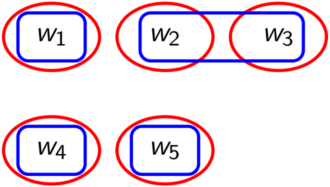

Note that $f(w_1, v_2) = w_2$ and $f(w_1, v_3) = w_4$, so there is common knowledge of S-rationality at w_1.

Aumann's Theorem: If Γ is a non-degenerate game of perfect information, then in all models of Γ, we have $C(A - Rat) \subseteq BI$

Stalnaker's Theorem: There exists a non-degenerate game Γ of perfect information and an extended model of Γ in which the selection function satisfies F1-F3 such that $C(S - Rat) \not\subseteq BI$.

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Revising beliefs during play:

"Although it is common knowledge that Ann would play across if v_3 were reached, if Ann were to play across at v_1, Bob would consider it possible that Ann would play down at v_3"

F4. For all players i and vertices v, if $w' \in [f(w, v)]_i$ then there exists a state $w'' \in [w]_i$ such that $\sigma(w')$ and $\sigma(w'')$ agree on the subtree of Γ below v.

Theorem (Halpern). If Γ is a non-degenerate game of perfect information, then for every extended model of Γ in which the selection function satisfies F1-F4, we have $C(S - Rat) \subseteq BI$. Moreover, there is an extend model of Γ in which the selection function satisfies F1-F4.

J. Halpern. *Substantive Rationality and Backward Induction*. Games and Economic Behavior, 37, pp. 425-435, 1998.