Logic and Artificial Intelligence

Lecture 4

Eric Pacuit

Currently Visiting the Center for Formal Epistemology, CMU

Center for Logic and Philosophy of Science
Tilburg University

ai.stanford.edu/∼epacuit
e.j.pacuit@uvt.nl

September 12, 2011

Quick Review: Basics of Modal Logic

Kripke Structures

Kripke Structures

The main idea:

- ‘It is sunny outside’ is currently true

Kripke Structures

The main idea:

- ‘It is sunny outside’ is currently true, but it is not necessary (for example, if we were currently in Amsterdam).

Kripke Structures

The main idea:

- 'It is sunny outside' is currently true, but it is not necessary (for example, if we were currently in Amsterdam).

- We say P is **necessary** provided P is true in all (relevant) situations (states, worlds, possibilities).

Kripke Structures

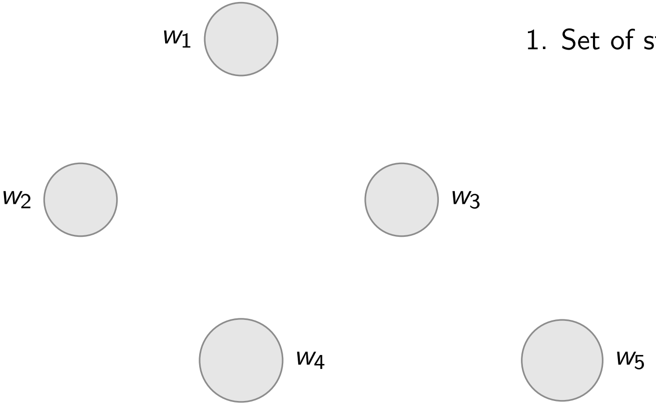
The main idea:

- 'It is sunny outside' is currently true, but it is not necessary (for example, if we were currently in Amsterdam).

- We say P is **necessary** provided P is true in all (relevant) situations (states, worlds, possibilities).

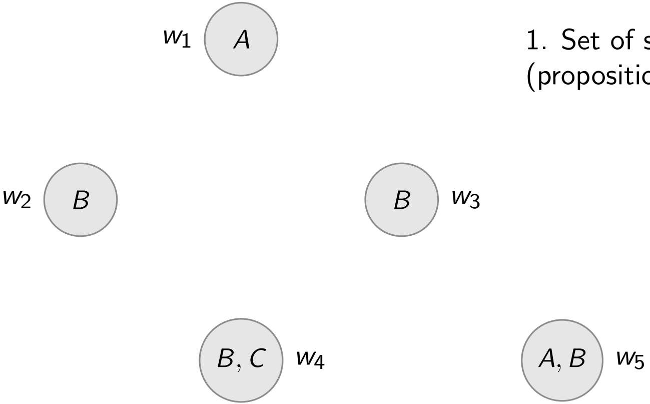
- A **Kripke structure** is
  1. A set of states, or worlds (each world specifies the truth value of all propositional variables)
  2. A **relation** on the set of states (specifying the "relevant situations")

A Kripke Structure



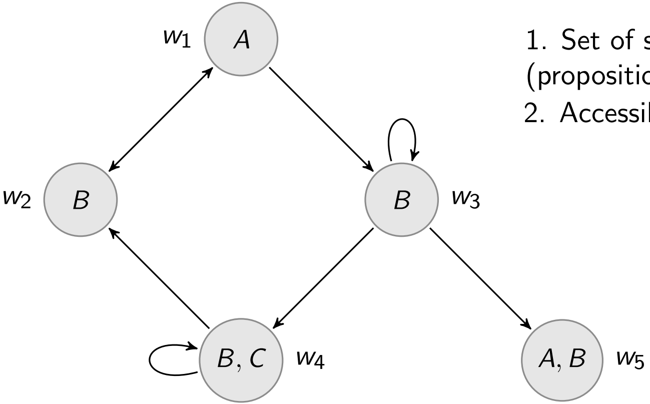
1. Set of states

A Kripke Structure



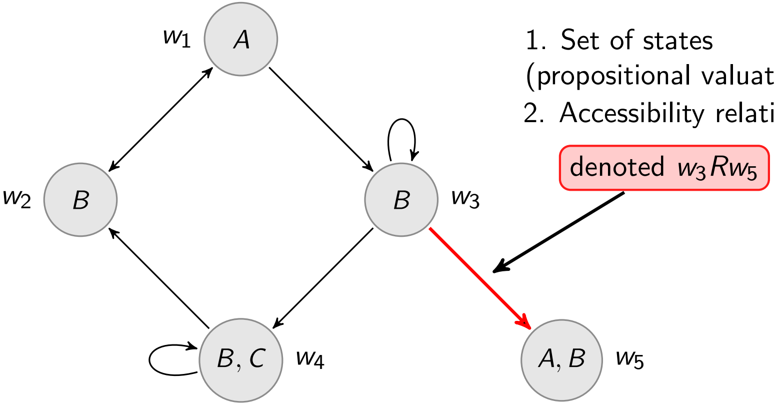
1. Set of states (propositional valuations)

A Kripke Structure



1. Set of states (propositional valuations)
2. Accessibility relation

A Kripke Structure



1. Set of states (propositional valuations)
2. Accessibility relation

Truth of Modal Formulas

We interpret formulas at states in a Kripke structure: $w \models P$ means P is true at state w.

Truth of Modal Formulas

We interpret formulas at states in a Kripke structure: $w \models P$ means P is true at state w.

We write wRv is v is accessible from state w.

Truth of Modal Formulas

We interpret formulas at states in a Kripke structure: $w \models P$ means P is true at state w.

We write wRv is v is accessible from state w.

1. $\Box P$ is true at state w iff P is true in all accessible worlds.

Truth of Modal Formulas

We interpret formulas at states in a Kripke structure: $w \models P$ means P is true at state w.

We write wRv is v is accessible from state w.

1. $\Box P$ is true at state w iff P is true in all accessible worlds.
   $w \models \Box P$ iff for all v, if wRv then $v \models P$

Truth of Modal Formulas

We interpret formulas at states in a Kripke structure: $w \models P$ means P is true at state w.

We write wRv is v is accessible from state w.

1. $\Box P$ is true at state w iff P is true in all accessible worlds.
   $w \models \Box P$ iff for all v, if wRv then $v \models P$
2. $\Diamond P$ is true at state w iff P is true at some accessible world.

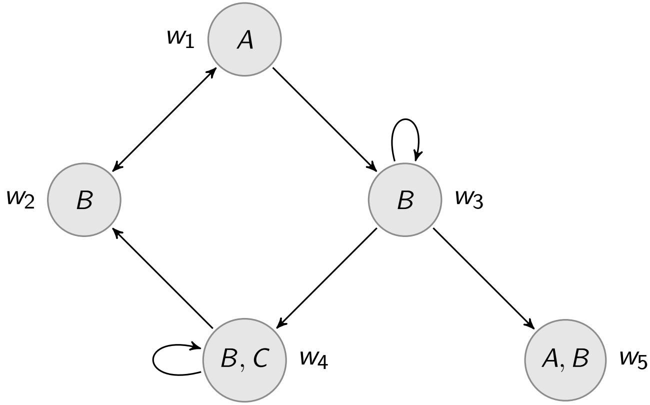
Truth of Modal Formulas

We interpret formulas at states in a Kripke structure: $w \models P$ means P is true at state w.

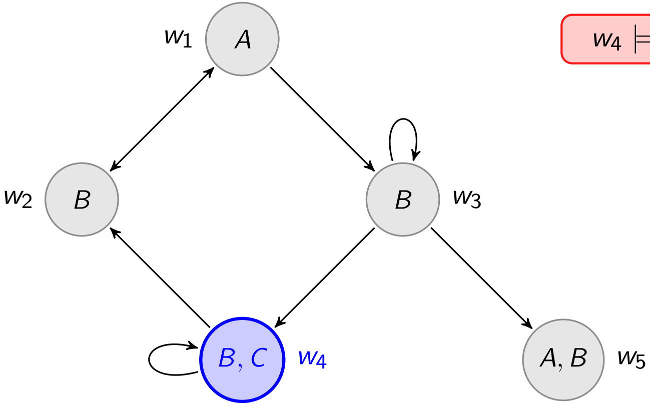
We write wRv is v is accessible from state w.

1. $\Box P$ is true at state w iff P is true in all accessible worlds.
   $w \models \Box P$ iff for all v, if wRv then $v \models P$
2. $\Diamond P$ is true at state w iff P is true at some accessible world.
   $w \models \Diamond P$ iff there exists v such that wRv and $v \models P$.

Example

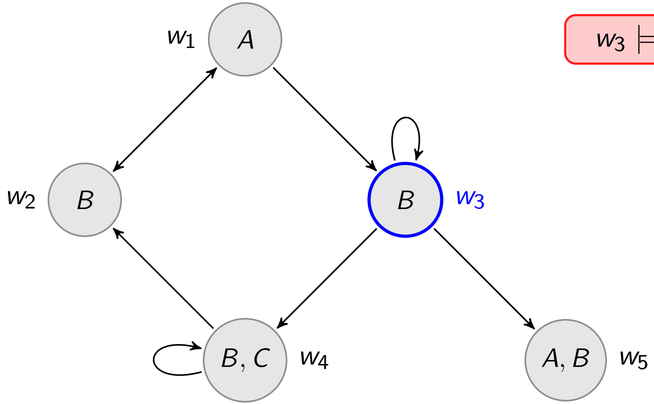


Example



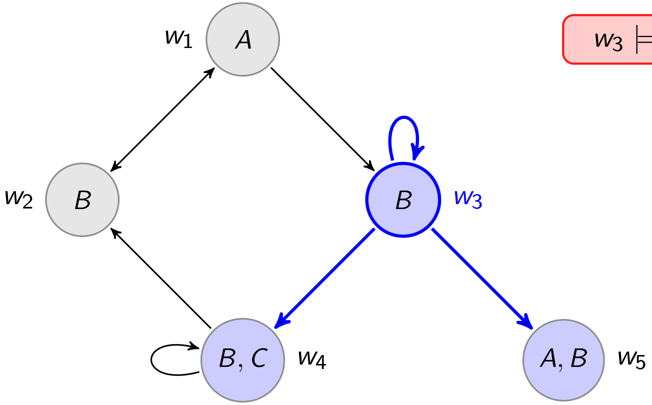
$w_4 \models B \wedge C$

Example



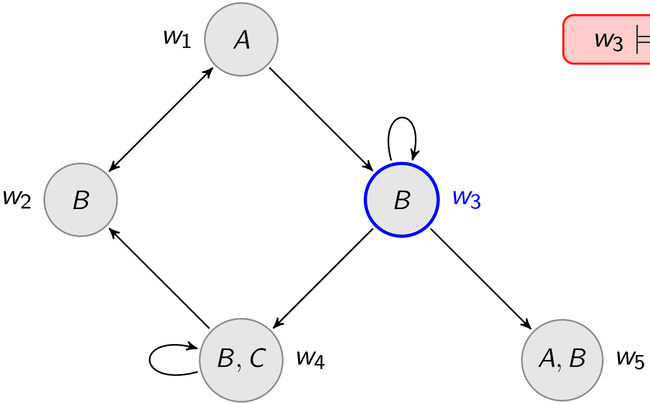
$w_3 \models \Box B$

Example



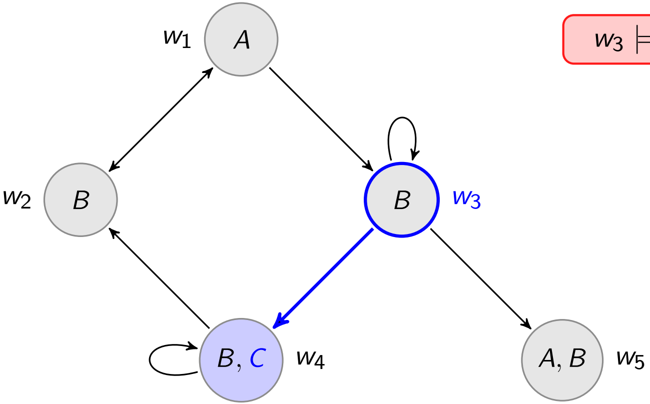
$w_3 \models \Box B$

Example



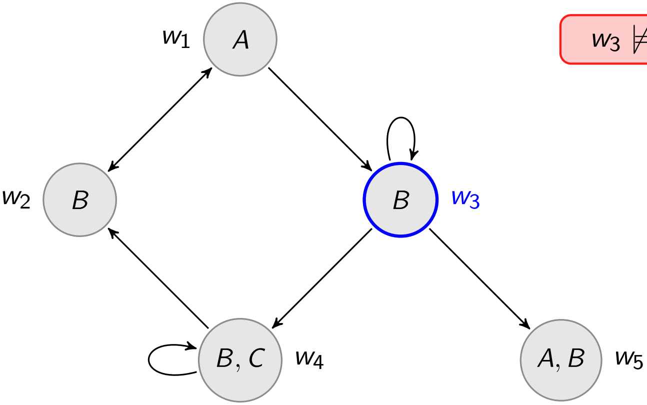
$w_3 \models \Diamond C$

Example



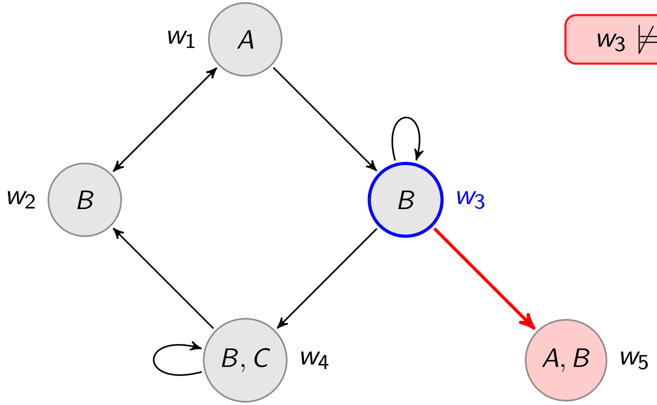
$w_3 \models \Diamond C$

Example



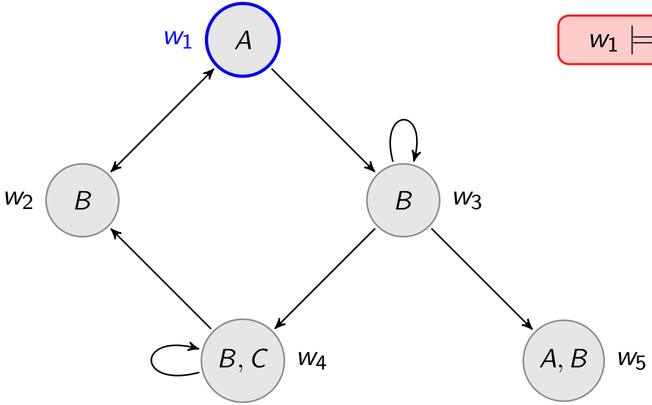
$w_3 \not\models \Box C$

Example



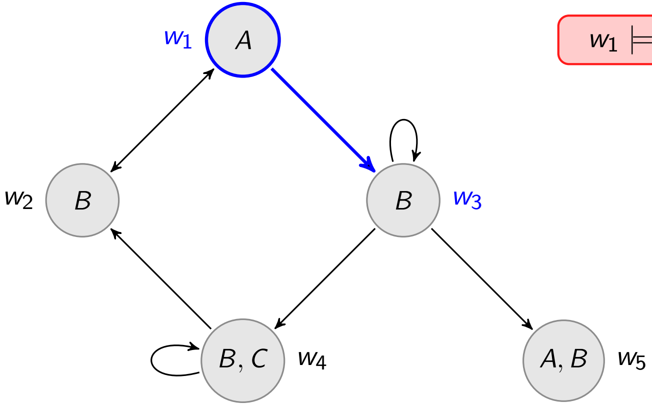
$w_3 \not\models \Box C$

Example



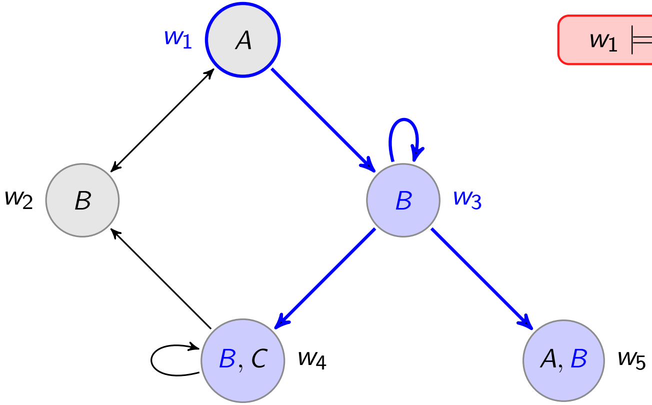
$w_1 \models \Diamond\Box B$

Example



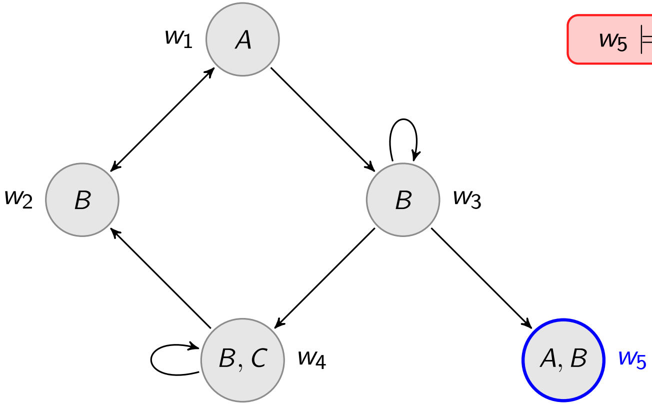
$w_1 \models \Diamond\Box B$

Example



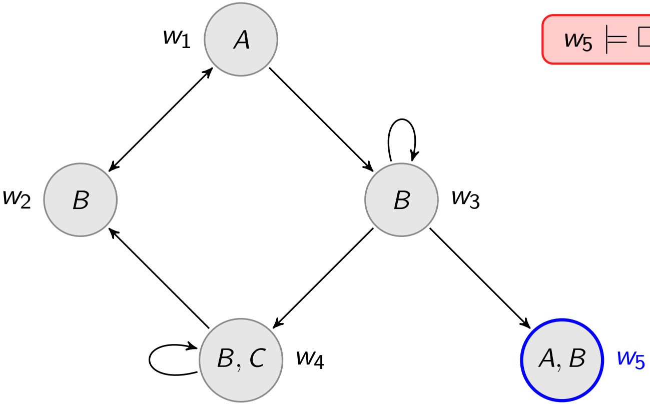
$w_1 \models \Diamond\Box B$

Example



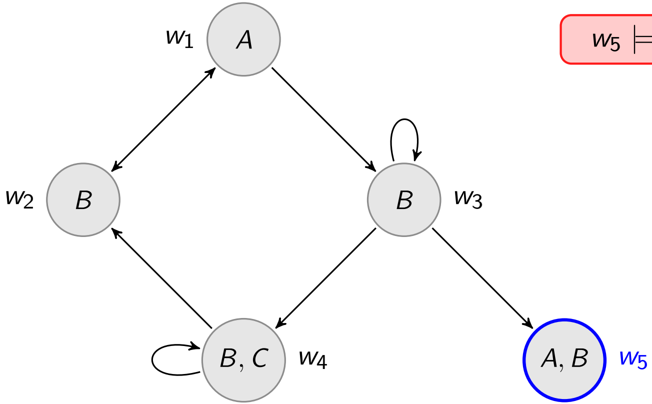
$w_5 \models \Box C$

Example

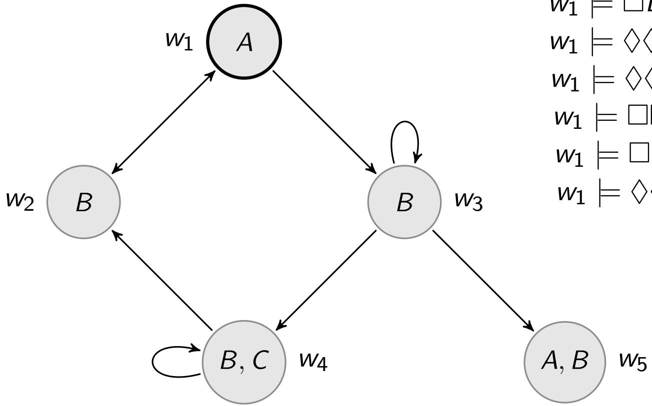


$w_5 \models \Box(B \land \neg B)$

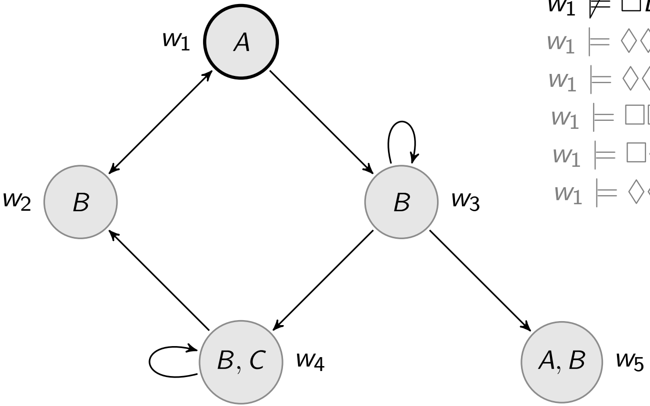
Example



$w_5 \models \neg\Diamond B$



$w_1 \models \Box B \wedge B$?
$w_1 \models \Diamond\Diamond B$?
$w_1 \models \Diamond\Diamond\Diamond B$?
$w_1 \models \Box\Box B$?
$w_1 \models \Box\Diamond C$?
$w_1 \models \Diamond\Diamond C$?



$w_1 \not\models \Box B \wedge B$

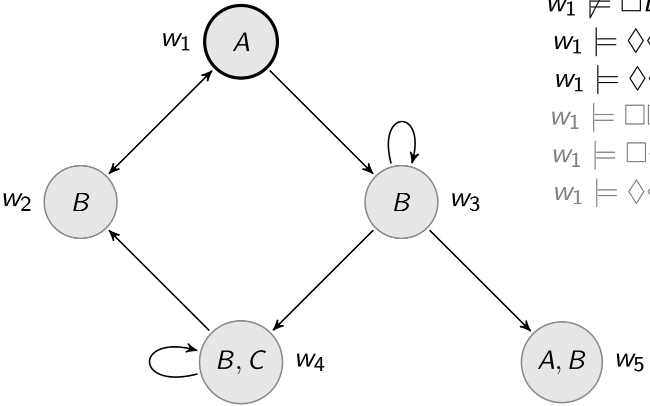
$w_1 \models \Diamond\Diamond B?$

$w_1 \models \Diamond\Diamond\Diamond B?$

$w_1 \models \Box\Box B?$

$w_1 \models \Box\Diamond C?$

$w_1 \models \Diamond\Diamond C?$



$w_1 \not\models \Box B \land B$

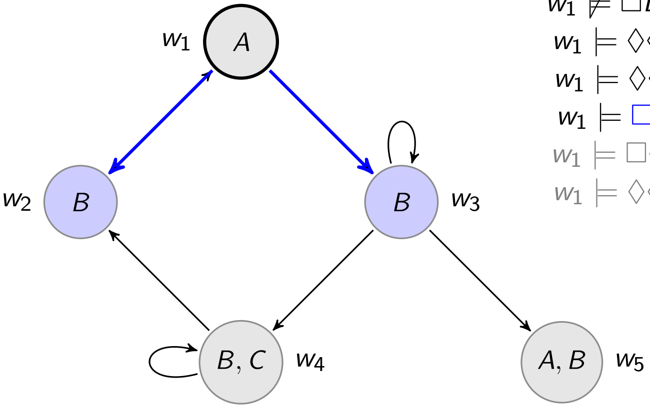
$w_1 \models \Diamond\Diamond B$

$w_1 \models \Diamond\Diamond\Diamond B$

$w_1 \models \Box\Box B?$

$w_1 \models \Box\Diamond C?$

$w_1 \models \Diamond\Diamond C?$



$w_1 \not\models \Box B \wedge B$

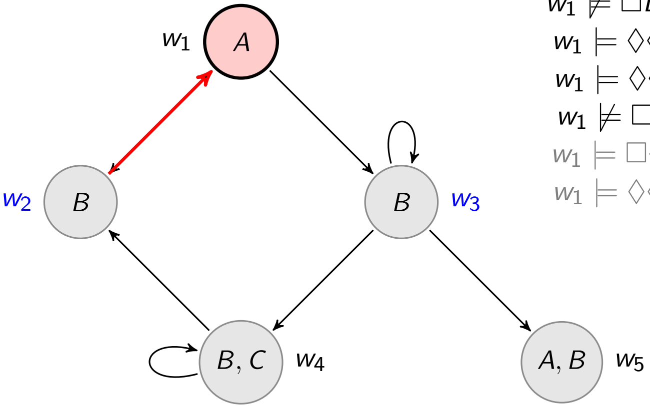
$w_1 \models \Diamond\Diamond B$

$w_1 \models \Diamond\Diamond\Diamond B$

$w_1 \models \Box\Box B$

$w_1 \models \Box\Diamond C?$

$w_1 \models \Diamond\Diamond C?$



$w_1 \not\models \Box B \wedge B$

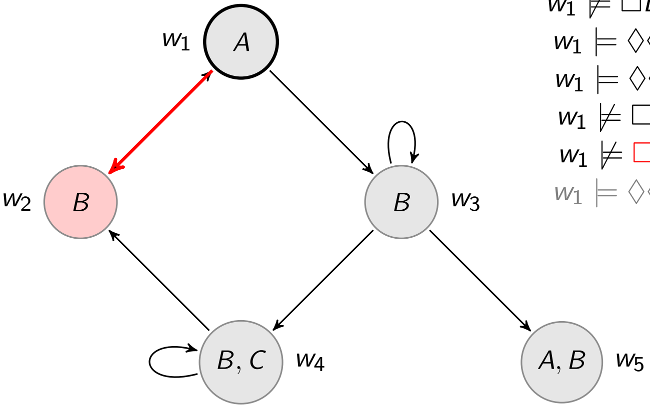
$w_1 \models \Diamond\Diamond B$

$w_1 \models \Diamond\Diamond\Diamond B$

$w_1 \not\models \Box\Box B$

$w_1 \models \Box\Diamond C?$

$w_1 \models \Diamond\Diamond C?$



$w_1 \not\models \Box B \wedge B$

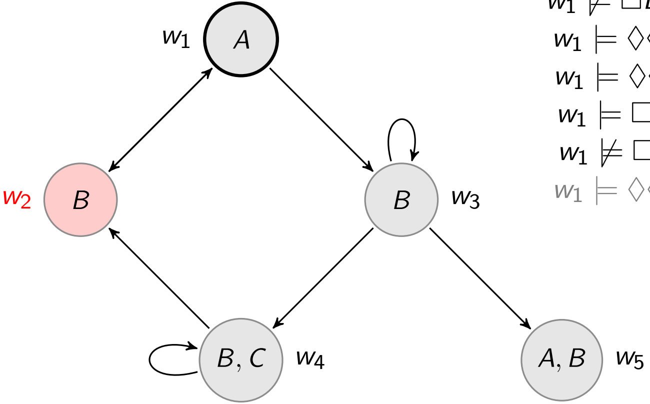
$w_1 \models \Diamond\Diamond B$

$w_1 \models \Diamond\Diamond\Diamond B$

$w_1 \not\models \Box\Box B$

$w_1 \not\models \Box\Diamond C$

$w_1 \models \Diamond\Diamond C?$



$w_1 \not\models \Box B \wedge B$

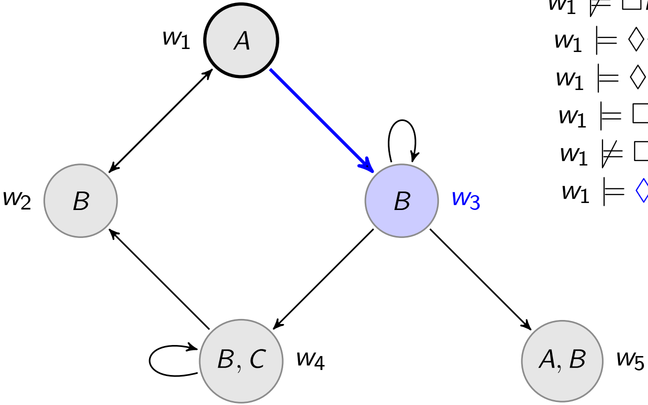
$w_1 \models \Diamond\Diamond B$

$w_1 \models \Diamond\Diamond\Diamond B$

$w_1 \models \Box\Box B$

$w_1 \not\models \Box\Diamond C$

$w_1 \models \Diamond\Diamond C$?



$w_1 \not\models \Box B \wedge B$

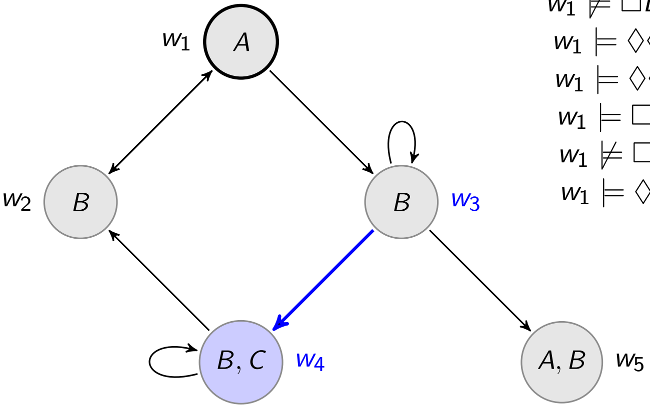
$w_1 \models \Diamond\Diamond B$

$w_1 \models \Diamond\Diamond\Diamond B$

$w_1 \models \Box\Box B$

$w_1 \not\models \Box\Diamond C$

$w_1 \models \Diamond\Diamond C$



$w_1 \not\models \Box B \wedge B$
$w_1 \models \Diamond\Diamond B$
$w_1 \models \Diamond\Diamond\Diamond B$
$w_1 \models \Box\Box B$
$w_1 \not\models \Box\Diamond C$
$w_1 \models \Diamond\Diamond C$

Some Facts

- $\Box P \vee \neg\Box P$ is always true (i.e., true at any state in any Kripke structure), but what about $\Box P \vee \Box\neg P$?

Some Facts

- $\Box P \vee \neg\Box P$ is always true (i.e., true at any state in any Kripke structure), but what about $\Box P \vee \Box\neg P$?
- $\Box P \wedge \Box Q \rightarrow \Box(P \wedge Q)$ is true at any state in any Kripke structure.

Some Facts

- $\Box P \vee \neg \Box P$ is always true (i.e., true at any state in any Kripke structure), but what about $\Box P \vee \Box \neg P$?

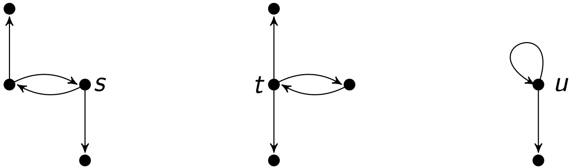
- $\Box P \wedge \Box Q \to \Box (P \wedge Q)$ is true at any state in any Kripke structure. What about $\Box (P \vee Q) \to \Box P \vee \Box Q$?

Some Facts

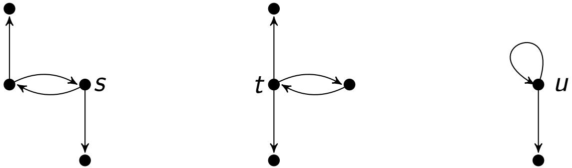
- $\Box P \vee \neg\Box P$ is always true (i.e., true at any state in any Kripke structure), but what about $\Box P \vee \Box\neg P$?
- $\Box P \wedge \Box Q \rightarrow \Box(P \wedge Q)$ is true at any state in any Kripke structure. What about $\Box(P \vee Q) \rightarrow \Box P \vee \Box Q$?
- $\Box P \leftrightarrow \neg\Diamond\neg P$ is true at any state in any Kripke structure.

Something to think about....

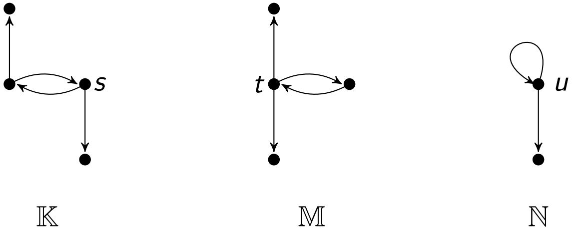
Which pair of states cannot be distinguished by a modal formula?
What about a first order formula?



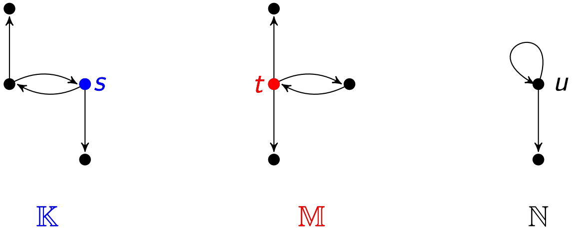
$\mathbb{K}$ $\mathbb{M}$ $\mathbb{N}$



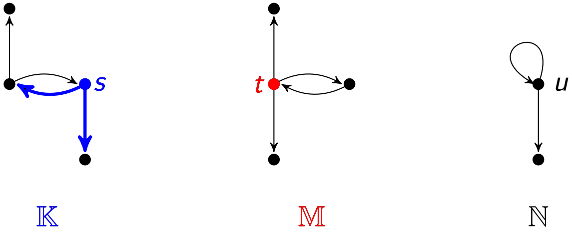
$\mathbb{K}$ $\mathbb{M}$ $\mathbb{N}$



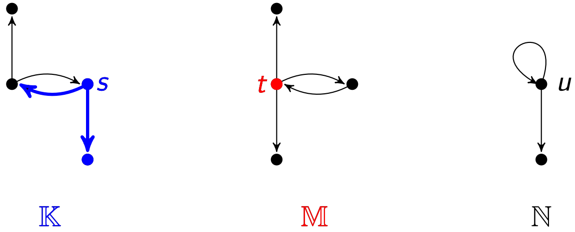
$$\Box(\Box\bot \vee \Diamond\Box\bot)$$



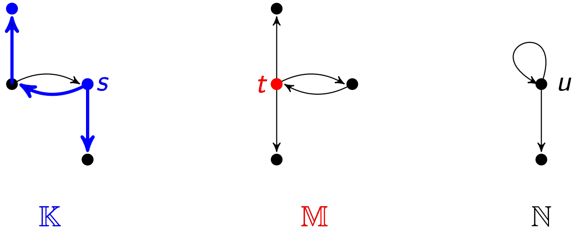
$$\Box(\Box\bot \vee \Diamond\Box\bot)$$



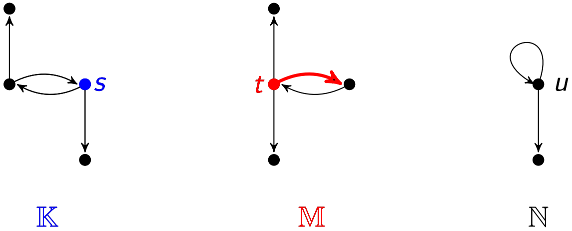
$\Box(\Box\bot \vee \Diamond\Box\bot)$



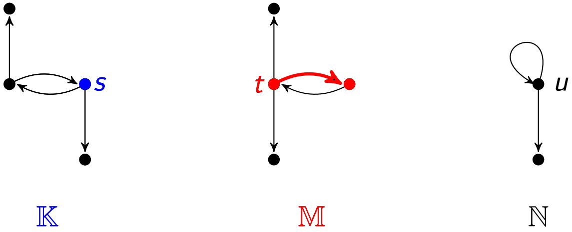
$\Box(\Box\bot \vee \Diamond\Box\bot)$



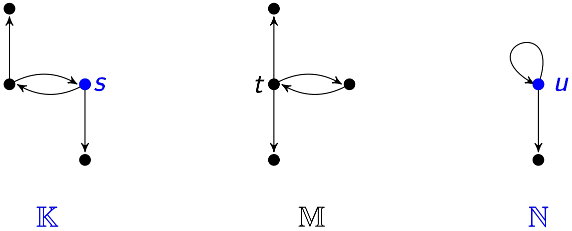
$\Box(\Box\bot \vee \Diamond\Box\bot)$



$\Box(\Box\bot \vee \Diamond\Box\bot)$



$\Box(\Box\bot \vee \Diamond\Box\bot)$


s
t
u
𝕂
𝕄
ℕ

Epistemic Logic

Write '$K\varphi$' for '$\Box\varphi$' with the intended interpretation "The agents knows that φ (is true)"

The Language: $\varphi \ := \ p \mid \neg\varphi \mid \varphi \wedge \psi \mid K\varphi$

Kripke Models: $\mathcal{M} = \langle W, R, V \rangle$ and $w \in W$

Truth: $\mathcal{M}, w \models \varphi$ is defined as follows:

- $\mathcal{M}, w \models p$ iff $w \in V(p)$ (with $p \in \mathrm{At}$)
- $\mathcal{M}, w \models \neg\varphi$ if $\mathcal{M}, w \not\models \varphi$
- $\mathcal{M}, w \models \varphi \wedge \psi$ if $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$
- $\mathcal{M}, w \models K\varphi$ if for each $v \in W$, if wRv, then $\mathcal{M}, v \models \varphi$

Logical Omniscience

- φ is valid then $K\varphi$ is valid
- $K\varphi \wedge K\psi \rightarrow K(\varphi \wedge \psi)$ is valid on all Kripke frames
- If $\varphi \rightarrow \psi$ is valid then $K\varphi \rightarrow K\psi$ is valid
- $K(\varphi \rightarrow \psi) \rightarrow (K\varphi \rightarrow K\psi)$ is valid on all Kripke frames.
- $\varphi \leftrightarrow \psi$ is valid then $K\varphi \leftrightarrow K\psi$ is valid

Correspondence

Definition

A model formula φ corresponds to a property P (of a relation in a Kripke frame) provided

$$\mathcal{F} \models \varphi \text{ iff } \mathcal{F} \text{ has } P$$

Correspondence

Definition

A model formula φ corresponds to a property P (of a relation in a Kripke frame) provided

$$\mathcal{F} \models \varphi \text{ iff } \mathcal{F} \text{ has } P$$

Modal Formula	Property	Philosophical Assumption

Correspondence

Definition

A model formula φ corresponds to a property P (of a relation in a Kripke frame) provided

$$\mathcal{F} \models \varphi \text{ iff } \mathcal{F} \text{ has } P$$

Modal Formula	Property	Philosophical Assumption
$K(\varphi \to \psi) \to (K\varphi \to K\psi)$	—	Logical Omniscience

Correspondence

Definition

A model formula φ corresponds to a property P (of a relation in a Kripke frame) provided

$$\mathcal{F} \models \varphi \text{ iff } \mathcal{F} \text{ has } P$$

Modal Formula	Property	Philosophical Assumption
$K(\varphi \to \psi) \to (K\varphi \to K\psi)$	—	Logical Omniscience
$K\varphi \to \varphi$	Reflexive	Truth

Correspondence

Definition

A model formula φ corresponds to a property P (of a relation in a Kripke frame) provided

$$\mathcal{F} \models \varphi \text{ iff } \mathcal{F} \text{ has } P$$

Modal Formula	Property	Philosophical Assumption
$K(\varphi \to \psi) \to (K\varphi \to K\psi)$	—	Logical Omniscience
$K\varphi \to \varphi$	Reflexive	Truth
$K\varphi \to KK\varphi$	Transitive	Positive Introspection

Correspondence

Definition

A model formula φ corresponds to a property P (of a relation in a Kripke frame) provided

$$\mathcal{F} \models \varphi \text{ iff } \mathcal{F} \text{ has } P$$

Modal Formula	Property	Philosophical Assumption
$K(\varphi \to \psi) \to (K\varphi \to K\psi)$	—	Logical Omniscience
$K\varphi \to \varphi$	Reflexive	Truth
$K\varphi \to KK\varphi$	Transitive	Positive Introspection
$\neg K\varphi \to K\neg K\varphi$	Euclidean	Negative Introspection

Correspondence

Definition

A model formula φ corresponds to a property P (of a relation in a Kripke frame) provided

$$\mathcal{F} \models \varphi \text{ iff } \mathcal{F} \text{ has } P$$

Modal Formula	Property	Philosophical Assumption
$K(\varphi \to \psi) \to (K\varphi \to K\psi)$	—	Logical Omniscience
$K\varphi \to \varphi$	Reflexive	Truth
$K\varphi \to KK\varphi$	Transitive	Positive Introspection
$\neg K\varphi \to K\neg K\varphi$	Euclidean	Negative Introspection
$\neg K\bot$	Serial	Consistency

The Logic **S5**

The logic **S5** contains the following axiom schemes and rules:

Pc	Axiomatization of Propositional Calculus
K	$K(\varphi \rightarrow \psi) \rightarrow (K\varphi \rightarrow K\psi)$
T	$K\varphi \rightarrow \varphi$
4	$K\varphi \rightarrow KK\varphi$
5	$\neg K\varphi \rightarrow K\neg K\varphi$
MP	$\dfrac{\varphi \qquad \varphi \rightarrow \psi}{\psi}$
Nec	$\dfrac{\varphi}{K\psi}$

The Logic **S5**

The logic **S5** contains the following axiom schemes and rules:

Pc	Axiomatization of Propositional Calculus
K	$K(\varphi \to \psi) \to (K\varphi \to K\psi)$
T	$K\varphi \to \varphi$
4	$K\varphi \to KK\varphi$
5	$\neg K\varphi \to K\neg K\varphi$
MP	$\dfrac{\varphi \qquad \varphi \to \psi}{\psi}$
Nec	$\dfrac{\varphi}{K\psi}$

Theorem

S5 *is sound and strongly complete with respect to the class of Kripke frames with equivalence relations.*

The Logic **S5**

The logic **S5** contains the following axiom schemes and rules:

Pc	Axiomatization of Propositional Calculus
K	$K(\varphi \to \psi) \to (K\varphi \to K\psi)$
T	$K\varphi \to \varphi$
4	$K\varphi \to KK\varphi$
5	$\neg K\varphi \to K\neg K\varphi$
MP	$\dfrac{\varphi \qquad \varphi \to \psi}{\psi}$
Nec	$\dfrac{\varphi}{K\psi}$

Theorem

S5 *is sound and strongly complete with respect to the class of Kripke frames with equivalence relations.*

The Logic **S5**

The logic **S5** contains the following axiom schemes and rules:

Pc	Axiomatization of Propositional Calculus
K	$K(\varphi \to \psi) \to (K\varphi \to K\psi)$
T	$K\varphi \to \varphi$
4	$K\varphi \to KK\varphi$
5	$\neg K\varphi \to K\neg K\varphi$
MP	$\dfrac{\varphi \qquad \varphi \to \psi}{\psi}$
Nec	$\dfrac{\varphi}{K\psi}$

Theorem

S5 *is sound and strongly complete with respect to the class of Kripke frames with equivalence relations.*

The Logic **S5**

The logic **S5** contains the following axiom schemes and rules:

Pc	Axiomatization of Propositional Calculus
K	$K(\varphi \to \psi) \to (K\varphi \to K\psi)$
T	$K\varphi \to \varphi$
4	$K\varphi \to KK\varphi$
5	$\neg K\varphi \to K\neg K\varphi$
MP	$\dfrac{\varphi \qquad \varphi \to \psi}{\psi}$
Nec	$\dfrac{\varphi}{K\psi}$

Theorem

S5 *is sound and strongly complete with respect to the class of Kripke frames with equivalence relations.*

The Logic **S5**

The logic **S5** contains the following axiom schemes and rules:

Pc	Axiomatization of Propositional Calculus
K	$K(\varphi \to \psi) \to (K\varphi \to K\psi)$
T	$K\varphi \to \varphi$
4	$K\varphi \to KK\varphi$
5	$\neg K\varphi \to K\neg K\varphi$
MP	$\dfrac{\varphi \qquad \varphi \to \psi}{\psi}$
Nec	$\dfrac{\varphi}{K\psi}$

Theorem

S5 *is sound and strongly complete with respect to the class of Kripke frames with equivalence relations.*

Multi-agent Epistemic Logic

The Language: $\varphi \ := \ p \mid \neg\varphi \mid \varphi \wedge \psi \mid K\varphi$

Kripke Models: $\mathcal{M} = \langle W, R, V\rangle$ and $w \in W$

Truth: $\mathcal{M}, w \models \varphi$ is defined as follows:

- $\mathcal{M}, w \models p$ iff $w \in V(p)$ (with $p \in \text{At}$)
- $\mathcal{M}, w \models \neg\varphi$ if $\mathcal{M}, w \not\models \varphi$
- $\mathcal{M}, w \models \varphi \wedge \psi$ if $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$
- $\mathcal{M}, w \models K\varphi$ if for each $v \in W$, if wRv, then $\mathcal{M}, v \models \varphi$

Multi-agent Epistemic Logic

The Language: $\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \psi \mid K_i\varphi$ with $i \in \mathcal{A}$

Kripke Models: $\mathcal{M} = \langle W, \{R_i\}_{i\in\mathcal{A}}, V\rangle$ and $w \in W$

Truth: $\mathcal{M}, w \models \varphi$ is defined as follows:

- $\mathcal{M}, w \models p$ iff $w \in V(p)$ (with $p \in \mathrm{At}$)
- $\mathcal{M}, w \models \neg\varphi$ if $\mathcal{M}, w \not\models \varphi$
- $\mathcal{M}, w \models \varphi \wedge \psi$ if $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$
- $\mathcal{M}, w \models K_i\varphi$ if for each $v \in W$, if wR_iv, then $\mathcal{M}, v \models \varphi$

Multi-agent Epistemic Logic

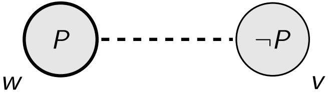
- $K_A K_B \varphi$: "Ann knows that Bob knows φ"
- $K_A(K_B\varphi \vee K_B\neg\varphi)$: "Ann knows that Bob knows whether φ
- $\neg K_B K_A K_B(\varphi)$: "Bob does not know that Ann knows that Bob knows that φ"

Where can we go from here?

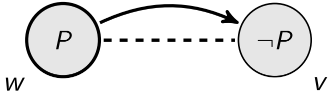
- More about the logic (complexity results, model checking, Tableaux systems, etc.)
- Additional informational attitudes (belief, certainty, justification, only knowing, awareness, etc.)
- Group notions
- Dynamics/temporal aspects

Where can we go from here?

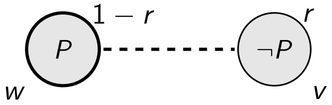
- More about the logic (complexity results, model checking, Tableaux systems, etc.)
- Additional informational attitudes (belief, certainty, justification, only knowing, awareness, etc.)
- Group notions
- Dynamics/temporal aspects



Ann does not know that P



Ann does not know that P, but she believes that $\neg P$



Ann does not know that P, but she believes that $\neg P$ is true to degree r.

Doxastic Logic

We write '$B\varphi$' for '$\Box\varphi$' with the intended interpretation "the agent believes that φ (is true)"

Conceptions of Belief

Binary: "all-out" belief. For any statement p, the agent either does or does not believe p. It is natural to take an *unqualified* assertion as a statement of belief of the speaker.

Conceptions of Belief

Binary: "all-out" belief. For any statement *p*, the agent either does or does not believe *p*. It is natural to take an *unqualified* assertion as a statement of belief of the speaker.

Graded: beliefs come in degrees. We are *more confident* in some of our beliefs than in others.

Conceptions of Belief

Binary: "all-out" belief. For any statement p, the agent either does or does not believe p. It is natural to take an *unqualified* assertion as a statement of belief of the speaker.

Graded: beliefs come in degrees. We are *more confident* in some of our beliefs than in others.

Eric Schwitzgebel. *Belief*. In The Stanford Encyclopedia of Philosophy.

Franz Huber. *Formal Theories of Belief*. In The Stanford Encyclopedia of Philosophy.

Conceptions of Beliefs: Questions

1. Are both conceptions of beliefs *reasonable*?

Conceptions of Beliefs: Questions

1. Are both conceptions of beliefs *reasonable*?
2. Is there a *unified* account?

Conceptions of Beliefs: Questions

1. Are both conceptions of beliefs *reasonable*?
2. Is there a *unified* account?

- Yes: Graded belief is an all-out belief in an "objective probability".

Conceptions of Beliefs: Questions

1. Are both conceptions of beliefs *reasonable*?
2. Is there a *unified* account?

- Yes: Graded belief is an all-out belief in an "objective probability".
- Yes: All-out belief is a special type of graded belief (eg., above a threshold $0 < t < 1$, probability 1).

Conceptions of Beliefs: Questions

1. Are both conceptions of beliefs *reasonable*?
2. Is there a *unified* account?

- Yes: Graded belief is an all-out belief in an "objective probability".
- Yes: All-out belief is a special type of graded belief (eg., above a threshold $0 < t < 1$, probability 1).
- No: Neither is a special case or species of the other.

Conceptions of Beliefs: Questions

1. Are both conceptions of beliefs *reasonable*?
2. Is there a *unified* account?
   - Yes: Graded belief is an all-out belief in an "objective probability".
   - Yes: All-out belief is a special type of graded belief (eg., above a threshold $0 < t < 1$, probability 1).
   - No: Neither is a special case or species of the other.
3. What are the *formal constraints* on rational belief?

Conceptions of Beliefs: Questions

1. Are both conceptions of beliefs *reasonable*?
2. Is there a *unified* account?
   - Yes: Graded belief is an all-out belief in an "objective probability".
   - Yes: All-out belief is a special type of graded belief (eg., above a threshold $0 < t < 1$, probability 1).
   - No: Neither is a special case or species of the other.
3. What are the *formal constraints* on rational belief?
   - rational graded beliefs should obey the laws of probability

Conceptions of Beliefs: Questions

1. Are both conceptions of beliefs *reasonable*?
2. Is there a *unified* account?
   - Yes: Graded belief is an all-out belief in an "objective probability".
   - Yes: All-out belief is a special type of graded belief (eg., above a threshold $0 < t < 1$, probability 1).
   - No: Neither is a special case or species of the other.
3. What are the *formal constraints* on rational belief?
   - rational graded beliefs should obey the laws of probability
   - rational all-out beliefs should be consistent/deductively closed

Conceptions of Beliefs: Questions

1. Are both conceptions of beliefs *reasonable*?
2. Is there a *unified* account?
   - Yes: Graded belief is an all-out belief in an "objective probability".
   - Yes: All-out belief is a special type of graded belief (eg., above a threshold $0 < t < 1$, probability 1).
   - No: Neither is a special case or species of the other.
3. What are the *formal constraints* on rational belief?
   - rational graded beliefs should obey the laws of probability
   - rational all-out beliefs should be consistent/deductively closed
   - how should we justify these constraints?

Conceptions of Beliefs: Questions

1. Are both conceptions of beliefs *reasonable*?
2. Is there a *unified* account?
   - Yes: Graded belief is an all-out belief in an "objective probability".
   - Yes: All-out belief is a special type of graded belief (eg., above a threshold $0 < t < 1$, probability 1).
   - No: Neither is a special case or species of the other.
3. What are the *formal constraints* on rational belief?
   - rational graded beliefs should obey the laws of probability
   - rational all-out beliefs should be consistent/deductively closed
   - how should we justify these constraints?

D. Christensen. *Putting Logic in its Place*. Oxford University Press.

H. Leitgeb. *The Lockean Thesis Revisited*. Working Paper, 2010.

The Logic **KD45**

The logic **KD45** contains the following axiom schemes and rules:

Pc	Axiomatization of Propositional Calculus
K	$B(\varphi \to \psi) \to (B\varphi \to B\psi)$
D	$B\varphi \to \neg B \neg \varphi$
4	$B\varphi \to BB\varphi$
5	$\neg B\varphi \to B\neg B\varphi$
MP	$\dfrac{\varphi \qquad \varphi \to \psi}{\psi}$
Nec	$\dfrac{\varphi}{B\psi}$

The Logic **KD45**

The logic **KD45** contains the following axiom schemes and rules:

Pc	Axiomatization of Propositional Calculus
K	$B(\varphi \rightarrow \psi) \rightarrow (B\varphi \rightarrow B\psi)$
D	$B\varphi \rightarrow \neg B \neg \varphi$
4	$B\varphi \rightarrow BB\varphi$
5	$\neg B\varphi \rightarrow B\neg B\varphi$
MP	$\dfrac{\varphi \qquad \varphi \rightarrow \psi}{\psi}$
Nec	$\dfrac{\varphi}{B\psi}$

Theorem

KD45 *is sound and strongly complete with respect to the class of Kripke frames where the relations are serial, transitive and Euclidean.*

Combining Logics of Knowledge and Belief

$\mathcal{M} = \langle W, \{\sim_i\}_{i \in \mathcal{A}}, \{R_i\}_{i \in \mathcal{A}}, V \rangle$ where

- $W \neq \emptyset$ is a set of states;
- each $\sim_i$ is an equivalence relation on W;
- each R_i is a serial, transitive, Euclidean relation on W; and
- V is a valuation function.

Combining Logics of Knowledge and Belief

$\mathcal{M} = \langle W, \{\sim_i\}_{i \in \mathcal{A}}, \{R_i\}_{i \in \mathcal{A}}, V \rangle$ where

- $W \neq \emptyset$ is a set of states;
- each $\sim_i$ is an equivalence relation on W;
- each R_i is a serial, transitive, Euclidean relation on W; and
- V is a valuation function.

What is the relationship between knowledge (K_i) and believe (B_i)?

- Each K_i is **S5**

Combining Logics of Knowledge and Belief

$\mathcal{M} = \langle W, \{\sim_i\}_{i \in \mathcal{A}}, \{R_i\}_{i \in \mathcal{A}}, V \rangle$ where

- $W \neq \emptyset$ is a set of states;
- each $\sim_i$ is an equivalence relation on W;
- each R_i is a serial, transitive, Euclidean relation on W; and
- V is a valuation function.

What is the relationship between knowledge (K_i) and believe (B_i)?

- Each K_i is **S5**
- Each B_i is **KD45**

Combining Logics of Knowledge and Belief

$\mathcal{M} = \langle W, \{\sim_i\}_{i \in \mathcal{A}}, \{R_i\}_{i \in \mathcal{A}}, V \rangle$ where

- $W \neq \emptyset$ is a set of states;
- each $\sim_i$ is an equivalence relation on W;
- each R_i is a serial, transitive, Euclidean relation on W; and
- V is a valuation function.

What is the relationship between knowledge (K_i) and believe (B_i)?

- Each K_i is **S5**
- Each B_i is **KD45**
- $K_i\varphi \rightarrow B_i\varphi$? "knowledge implies belief"

Combining Logics of Knowledge and Belief

$\mathcal{M} = \langle W, \{\sim_i\}_{i \in \mathcal{A}}, \{R_i\}_{i \in \mathcal{A}}, V \rangle$ where

- $W \neq \emptyset$ is a set of states;
- each $\sim_i$ is an equivalence relation on W;
- each R_i is a serial, transitive, Euclidean relation on W; and
- V is a valuation function.

What is the relationship between knowledge (K_i) and believe (B_i)?

- Each K_i is **S5**
- Each B_i is **KD45**
- $K_i\varphi \to B_i\varphi$? "knowledge implies belief"
- $B_i\varphi \to B_i K_i\varphi$? "positive certainty"

Combining Logics of Knowledge and Belief

$\mathcal{M} = \langle W, \{\sim_i\}_{i \in \mathcal{A}}, \{R_i\}_{i \in \mathcal{A}}, V \rangle$ where

- $W \neq \emptyset$ is a set of states;
- each $\sim_i$ is an equivalence relation on W;
- each R_i is a serial, transitive, Euclidean relation on W; and
- V is a valuation function.

What is the relationship between knowledge (K_i) and believe (B_i)?

- Each K_i is **S5**
- Each B_i is **KD45**
- $K_i\varphi \to B_i\varphi$? "knowledge implies belief"
- $B_i\varphi \to B_iK_i\varphi$? "positive certainty"
- $B_i\varphi \to K_iB_i\varphi$?

An Issue

- Suppose that p is something you are certain of (you *believe* it with probability one), but is false: $\neg p \wedge Bp$

An Issue

- Suppose that p is something you are certain of (you *believe* it with probability one), but is false: $\neg p \wedge Bp$
- Assuming 1. B satisfies **KD45**, 2. K satisfies **S5**, 3. knowledge implies believe and 4. positive certainty leads to a contradiction.

An Issue

- Suppose that p is something you are certain of (you *believe* it with probability one), but is false: $\neg p \wedge Bp$
- Assuming 1. B satisfies **KD45**, 2. K satisfies **S5**, 3. knowledge implies believe and 4. positive certainty leads to a contradiction.
- $Bp \rightarrow BKp$

An Issue

- Suppose that p is something you are certain of (you *believe* it with probability one), but is false: $\neg p \wedge Bp$
- Assuming 1. B satisfies **KD45**, 2. K satisfies **S5**, 3. knowledge implies believe and 4. positive certainty leads to a contradiction.
- $Bp \rightarrow BKp$
- $\neg p \rightarrow \neg Kp$

An Issue

- Suppose that p is something you are certain of (you *believe* it with probability one), but is false: $\neg p \wedge Bp$
- Assuming 1. B satisfies **KD45**, 2. K satisfies **S5**, 3. knowledge implies believe and 4. positive certainty leads to a contradiction.
- $Bp \rightarrow BKp$
- $\neg p \rightarrow \neg Kp \rightarrow K\neg Kp$

An Issue

- Suppose that p is something you are certain of (you *believe* it with probability one), but is false: $\neg p \wedge Bp$
- Assuming 1. B satisfies **KD45**, 2. K satisfies **S5**, 3. knowledge implies believe and 4. positive certainty leads to a contradiction.
- $Bp \rightarrow BKp$
- $\neg p \rightarrow \neg Kp \rightarrow K\neg Kp \rightarrow B\neg Kp$

An Issue

- Suppose that p is something you are certain of (you *believe* it with probability one), but is false: $\neg p \wedge Bp$
- Assuming 1. B satisfies **KD45**, 2. K satisfies **S5**, 3. knowledge implies believe and 4. positive certainty leads to a contradiction.
- $Bp \rightarrow BKp$
- $\neg p \rightarrow \neg Kp \rightarrow K\neg Kp \rightarrow B\neg Kp$
- So, $BKp \wedge B\neg Kp$ also holds, but this contradictions $B\varphi \rightarrow \neg B\neg\varphi$.

J. Halpern. *Should Knowledge Entail Belief?*. Journal of Philosophical Logic, 25:5, 1996, pp. 483-494.

Adding Beliefs

Epistemic Models: $\mathcal{M} = \langle W, \{\sim_i\}_{i \in \mathcal{A}}, V \rangle$

Truth: $\mathcal{M}, w \models \varphi$ is defined as follows:

- $\mathcal{M}, w \models p$ iff $w \in V(p)$ (with $p \in \text{At}$)
- $\mathcal{M}, w \models \neg\varphi$ if $\mathcal{M}, w \not\models \varphi$
- $\mathcal{M}, w \models \varphi \wedge \psi$ if $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$
- $\mathcal{M}, w \models K_i\varphi$ if for each $v \in W$, if $w \sim_i v$, then $\mathcal{M}, v \models \varphi$

Adding Beliefs

Epistemic-Doxastic Models: $\mathcal{M} = \langle W, \{\sim_i\}_{i\in\mathcal{A}}, \{\preceq_i\}_{i\in\mathcal{A}}, V\rangle$

Truth: $\mathcal{M}, w \models \varphi$ is defined as follows:

- $\mathcal{M}, w \models p$ iff $w \in V(p)$ (with $p \in \mathrm{At}$)
- $\mathcal{M}, w \models \neg\varphi$ if $\mathcal{M}, w \not\models \varphi$
- $\mathcal{M}, w \models \varphi \wedge \psi$ if $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$
- $\mathcal{M}, w \models K_i\varphi$ if for each $v \in W$, if $w{\sim_i}v$, then $\mathcal{M}, v \models \varphi$

Adding Beliefs

Epistemic-Doxastic Models: $\mathcal{M} = \langle W, \{\sim_i\}_{i\in\mathcal{A}}, \{\preceq_i\}_{i\in\mathcal{A}}, V\rangle$

Plausibility Relation: $\preceq_i \subseteq W \times W$. $w \preceq_i v$ means

"v is at least as plausibile as w."

Adding Beliefs

Epistemic-Doxastic Models: $\mathcal{M} = \langle W, \{\sim_i\}_{i\in\mathcal{A}}, \{\preceq_i\}_{i\in\mathcal{A}}, V\rangle$

Plausibility Relation: $\preceq_i \subseteq W \times W$. $w \preceq_i v$ means

"v is at least as plausibile as w."

Properties of $\preceq_i$: reflexive, transitive, and *well-founded*.

Adding Beliefs

Epistemic-Doxastic Models: $\mathcal{M} = \langle W, \{\sim_i\}_{i\in\mathcal{A}}, \{\preceq_i\}_{i\in\mathcal{A}}, V\rangle$

Plausibility Relation: $\preceq_i \subseteq W \times W$. $w \preceq_i v$ means

"v is at least as plausibile as w."

Properties of $\preceq_i$: reflexive, transitive, and *well-founded*.

Most Plausible: For $X \subseteq W$, let

$$Min_{\preceq_i}(X) = \{v \in W \mid v \preceq_i w \text{ for all } w \in X\}$$

Adding Beliefs

Epistemic-Doxastic Models: $\mathcal{M} = \langle W, \{\sim_i\}_{i\in\mathcal{A}}, \{\preceq_i\}_{i\in\mathcal{A}}, V\rangle$

Plausibility Relation: $\preceq_i \subseteq W \times W$. $w \preceq_i v$ means

"v is at least as plausibile as w."

Properties of $\preceq_i$: reflexive, transitive, and *well-founded*.

Most Plausible: For $X \subseteq W$, let

$$Min_{\preceq_i}(X) = \{v \in W \mid v \preceq_i w \text{ for all } w \in X\}$$

Assumptions:

1. *plausibility implies possibility*: if $w \preceq_i v$ then $w \sim_i v$.
2. *locally-connected*: if $w \sim_i v$ then either $w \preceq_i v$ or $v \preceq_i w$.

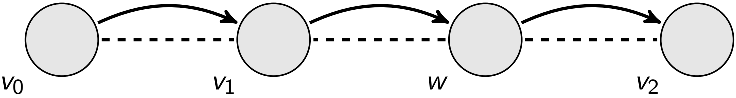
Adding Beliefs

Epistemic-Doxastic Models: $\mathcal{M} = \langle W, \{\sim_i\}_{i \in \mathcal{A}}, \{\preceq_i\}_{i \in \mathcal{A}}, V \rangle$

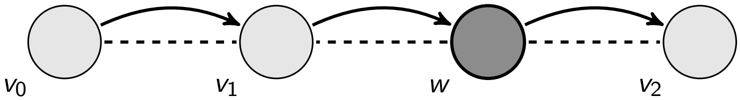
Truth: $\mathcal{M}, w \models \varphi$ is defined as follows:

- $\mathcal{M}, w \models p$ iff $w \in V(p)$ (with $p \in \text{At}$)
- $\mathcal{M}, w \models \neg\varphi$ if $\mathcal{M}, w \not\models \varphi$
- $\mathcal{M}, w \models \varphi \wedge \psi$ if $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$
- $\mathcal{M}, w \models K_i\varphi$ if for each $v \in W$, if $w \sim_i v$, then $\mathcal{M}, v \models \varphi$
- $\mathcal{M}, w \models B_i\varphi$ if for each $v \in Min_{\preceq_i}([w]_i)$, $\mathcal{M}, v \models \varphi$
  $[w]_i = \{v \mid w \sim_i v\}$ is the agent's **information cell**.

Grades of Doxastic Strength

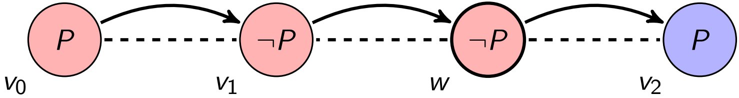


Grades of Doxastic Strength



Suppose that w is the current state.

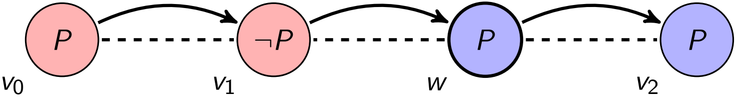
Grades of Doxastic Strength



Suppose that w is the current state.

- **Belief** (BP)

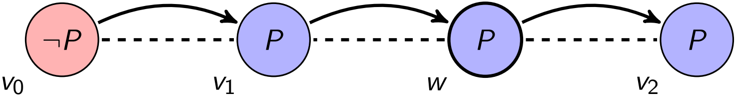
Grades of Doxastic Strength



Suppose that w is the current state.

- **Belief** (BP)
- **Safe Belief** ($\Box P$)

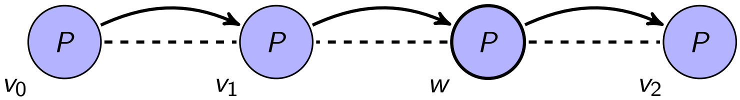
Grades of Doxastic Strength



Suppose that w is the current state.

- **Belief** (BP)
- **Safe Belief** ($\Box P$)
- **Strong Belief** ($B^s P$)

Grades of Doxastic Strength



Suppose that w is the current state.

- **Belief** (BP)
- **Safe Belief** ($\Box P$)
- **Strong Belief** ($B^s P$)
- **Knowledge** (KP)

Types of Beliefs: Logical Characterizations

- $\mathcal{M}, w \models K_i\varphi$ iff $\mathcal{M}, w \models B_i^{\psi}\varphi$ for all ψ
  i knows φ iff i continues to believe φ given any new information

Types of Beliefs: Logical Characterizations

- $\mathcal{M}, w \models K_i\varphi$ iff $\mathcal{M}, w \models B_i^{\psi}\varphi$ for all ψ
  i knows φ iff i continues to believe φ given any new information
- $\mathcal{M}, w \models \Box_i\varphi$ iff $\mathcal{M}, w \models B_i^{\psi}\varphi$ for all ψ with $\mathcal{M}, w \models \psi$.
  i safely beliefs φ iff i continues to believe φ given any true formula.

Types of Beliefs: Logical Characterizations

- $\mathcal{M}, w \models K_i\varphi$ iff $\mathcal{M}, w \models B_i^{\psi}\varphi$ for all ψ
  i knows φ iff *i* continues to believe φ given any new information
- $\mathcal{M}, w \models \Box_i\varphi$ iff $\mathcal{M}, w \models B_i^{\psi}\varphi$ for all ψ with $\mathcal{M}, w \models \psi$.
  i safely beliefs φ iff *i* continues to believe φ given any true formula.
- $\mathcal{M}, w \models B_i^s\varphi$ iff $\mathcal{M}, w \models B_i\varphi$ and $\mathcal{M}, w \models B_i^{\psi}\varphi$ for all ψ with $\mathcal{M}, w \models \neg K_i(\psi \rightarrow \neg\varphi)$.
  i strongly believes φ iff *i* believes φ and continues to believe φ given any evidence (truthful or not) that is not known to contradict φ.

Types of Beliefs: Logical Characterizations

- $\mathcal{M}, w \models K_i\varphi$ iff $\mathcal{M}, w \models B_i^{\psi}\varphi$ for all ψ
  i knows φ iff i continues to believe φ given any new information
- $\mathcal{M}, w \models \Box_i\varphi$ iff $\mathcal{M}, w \models B_i^{\psi}\varphi$ for all ψ with $\mathcal{M}, w \models \psi$.
  i safely beliefs φ iff i continues to believe φ given any true formula.
- $\mathcal{M}, w \models B_i^{s}\varphi$ iff $\mathcal{M}, w \models B_i\varphi$ and $\mathcal{M}, w \models B_i^{\psi}\varphi$ for all ψ with $\mathcal{M}, w \models \neg K_i(\psi \rightarrow \neg\varphi)$.
  i strongly believes φ iff i believes φ and continues to believe φ given any evidence (truthful or not) that is not known to contradict φ.

We can now **define** belief and strong belief:

- $B_i^{\varphi}\psi \ := \ L_i\varphi \rightarrow L_i(\varphi \wedge \Box_i(\varphi \rightarrow \psi))$
- $B_i^{s}\varphi \ := \ B_i\varphi \wedge K_i(\varphi \rightarrow \Box_i\varphi)$

A. Baltag and S. Smets. *Course notes on dynamic belief revision*. ESSLLI, NASSLLI,...

Conditional Beliefs

$B_i^\varphi\psi$: Agent i believes ψ, *given that* φ *is true*.

$\mathcal{M}, w \models B_i^\varphi\psi$ if for each $v \in Min_{\preceq_i}([w]_i \cap [\![\varphi]\!])$, $\mathcal{M}, v \models \varphi$
where $[\![\varphi]\!] = \{w \mid \mathcal{M}, w \models \varphi\}$

Conditional Beliefs

$B_i^{\varphi}\psi$: Agent i believes ψ, *given that* φ *is true*.

$\mathcal{M}, w \models B_i^{\varphi}\psi$ if for each $v \in Min_{\preceq_i}([w]_i \cap [\![\varphi]\!])$, $\mathcal{M}, v \models \varphi$
where $[\![\varphi]\!] = \{w \mid \mathcal{M}, w \models \varphi\}$

Is $B\varphi \to B^{\psi}\varphi$ valid?

Conditional Beliefs

$B_i^{\varphi}\psi$: Agent i believes ψ, *given that* φ *is true*.

$\mathcal{M}, w \models B_i^{\varphi}\psi$ if for each $v \in Min_{\preceq_i}([w]_i \cap [\![\varphi]\!])$, $\mathcal{M}, v \models \varphi$
where $[\![\varphi]\!] = \{w \mid \mathcal{M}, w \models \varphi\}$

Is $B\varphi \to B^{\psi}\varphi$ valid? No

Conditional Beliefs

$B_i^{\varphi}\psi$: Agent i believes ψ, *given that* φ *is true*.

$\mathcal{M}, w \models B_i^{\varphi}\psi$ if for each $v \in Min_{\preceq_i}([w]_i \cap [\![\varphi]\!])$, $\mathcal{M}, v \models \varphi$
where $[\![\varphi]\!] = \{w \mid \mathcal{M}, w \models \varphi\}$

Is $B\varphi \to B^{\psi}\varphi$ valid? No

What about $B\varphi \to B^{\psi}\varphi \vee B^{\neg\psi}\varphi$?

Conditional Beliefs

$B_i^{\varphi}\psi$: Agent i believes ψ, *given that* φ *is true*.

$\mathcal{M}, w \models B_i^{\varphi}\psi$ if for each $v \in Min_{\preceq_i}([w]_i \cap [\![\varphi]\!])$, $\mathcal{M}, v \models \varphi$
where $[\![\varphi]\!] = \{w \mid \mathcal{M}, w \models \varphi\}$

Is $B\varphi \rightarrow B^{\psi}\varphi$ valid? No

What about $B\varphi \rightarrow B^{\psi}\varphi \vee B^{\neg\psi}\varphi$? Yes (but need connectedness...)

Conditional Beliefs

$B_i^{\varphi}\psi$: Agent i believes ψ, *given that* φ *is true*.

$\mathcal{M}, w \models B_i^{\varphi}\psi$ if for each $v \in Min_{\preceq_i}([w]_i \cap [\![\varphi]\!])$, $\mathcal{M}, v \models \varphi$
where $[\![\varphi]\!] = \{w \mid \mathcal{M}, w \models \varphi\}$

Is $B\varphi \rightarrow B^{\psi}\varphi$ valid? No

What about $B\varphi \rightarrow B^{\psi}\varphi \vee B^{\neg\psi}\varphi$? Yes (but need connectedness...)

What does it mean if $B^{\neg\varphi}\bot$ is true at a state?

Conditional Beliefs

$B_i^{\varphi}\psi$: Agent i believes ψ, *given that* φ *is true*.

$\mathcal{M}, w \models B_i^{\varphi}\psi$ if for each $v \in Min_{\preceq_i}([w]_i \cap [\![\varphi]\!])$, $\mathcal{M}, v \models \varphi$
where $[\![\varphi]\!] = \{w \mid \mathcal{M}, w \models \varphi\}$

Is $B\varphi \rightarrow B^{\psi}\varphi$ valid? No

What about $B\varphi \rightarrow B^{\psi}\varphi \vee B^{\neg\psi}\varphi$? Yes (but need connectedness...)

What does it mean if $B^{\neg\varphi}\bot$ is true at a state? The agent knows φ

Conditional Beliefs

$\mathcal{M}, w \models B_i^{\varphi}\psi$ if for each $v \in Min_{\preceq_i}([w]_i \cap [\![\varphi]\!])$, $\mathcal{M}, v \models \varphi$
where $[\![\varphi]\!] = \{w \mid \mathcal{M}, w \models \varphi\}$

Core Logical Principles:

1. $B^{\varphi}\varphi$
2. $B^{\varphi}\psi \to B^{\varphi}(\psi \vee \chi)$
3. $(B^{\varphi}\psi_1 \wedge B^{\varphi}\psi_2) \to B^{\varphi}(\psi_1 \wedge \psi_2)$
4. $(B^{\varphi_1}\psi \wedge B^{\varphi_2}\psi) \to B^{\varphi_1 \vee \varphi_2}\psi$
5. $(B^{\varphi}\psi \wedge B^{\psi}\varphi) \to (B^{\varphi}\chi \leftrightarrow B^{\psi}\chi)$

J. Burgess. *Quick completeness proofs for some logics of conditionals*. *Notre Dame Journal of Formal Logic 22*, 76 – 84, 1981.

Next: Group Notions