Proof with improved bound

Given any Bayesian structure $\langle W, \{\sim_i\}_{i\in\mathcal{A}}, \{\pi_i\}_{i\in\mathcal{A}}\rangle$, we define

> i ***believes to degree at least*** p**:** $B_i^p : \wp(W) \to \wp(W)$ is defined as
> $B_i^p(E) = \{w \mid \pi_i(E \mid [w]_i) \geq p\}$

Fact 1 *Let $\langle W, \{\sim_i\}_{i\in\mathcal{A}}, \pi\rangle$ be a Bayesian model with a common prior. Then,*

1. *$B_i^p(E)$ is the union of elements from Π_i.*
2. *If E is the union of elements from Π_i, then $B_i^p(E) = E$.*
3. *$B_i^p(B_i^p(E)) = B_i^p(E)$*
4. *If $E \subseteq F$ then $B_i^p(E) \subseteq B_i^p(F)$*
5. *If (E_n) is a decreasing sequence of events then*

$$B_i^p\left(\bigcap_n E_n\right) = \bigcap_n B_i^p(E_n)$$

6. *$\pi(E \mid B_i^p(E)) \geq p$*

Definition 1 An event E is an **evident** p**-belief** if for each $i \in \mathcal{A}$, $E \subseteq B_i^p(E)$. The common p-belief operator $C^p : \wp(W) \to \wp(W)$ is defined as follows:

$$C^p(F) = \{w \mid \text{ there is an evident } p\text{-belief } E \text{ with } w \in E \text{ and } E \subseteq B_i^p(F) \text{ for each } i \in \mathcal{A}\}$$

◁

The following proof is based on the short proof from

Zvika Neeman (1996). Approximating Agreeing to Disagree Results with Common p-Beliefs, *Games and Economic Behavior*, **12**, pp. 162 - 164

Theorem 2 (Generalized Agreeing to Disagree Theorem) *Let $\langle W, \{\sim_i\}_{i\in\mathcal{A}}, \pi\rangle$ be a Bayesian model with a common prior. For any event $X \subseteq W$, if $C^p\left(\bigcap_{i\in\mathcal{A}} X_{i,r_i}\right) \neq \emptyset$ then for each $i, j \in \mathcal{A}$, $|r_i - r_j| \leq (1-p)$.*

Proof. Let $D = \bigcap_{i\in\mathcal{A}} X_{i,r_i}$. Suppose that $w \in C^p(D)$. Then there is an $E \subseteq W$ such that 1. $w \in E$, 2. $E \subseteq B_i^p(E)$ for each $i \in \mathcal{A}$ and 3. $E \subseteq B_i^p(D)$ for each $i \in \mathcal{A}$. Then $E \subseteq \bigcap_{i\in\mathcal{A}} B_i^p(E) = B^p(E)$. For each $i \in \mathcal{A}$, let $Z_i = B_i^p(E)$ and $Z = B^p(E)$.

Claim 1 $\pi(Z \mid Z_i) \geq p$

Proof of Claim 1. Since $E \subseteq \bigcap_{i \in \mathcal{A}} B_i^p(E) = B^p(E) = Z$, we have $Z_i = B_i^p(E) \subseteq B_i^p(Z)$. Hence, $\pi(Z_i) \leq \pi(B_i^p(Z))$ and so $\frac{1}{\pi(Z_i)} \geq \frac{1}{\pi(B_i^p(Z))}$. Furthermore, $Z \subseteq Z_i$, so $Z \cap Z_i = Z$.

$$\pi(Z \mid Z_i) = \frac{\pi(Z \cap Z_i)}{\pi(Z_i)} = \frac{\pi(Z)}{\pi(Z_i)} \geq \frac{\pi(Z)}{\pi(B_i^p(Z))} \geq \frac{\pi(Z \cap B_i^p(Z))}{\pi(B_i^p(Z))} = \pi(Z \mid B_i^p(Z)) \geq p$$

QED (of Claim)

Claim 2 $\pi(X \mid Z_i) = r_i$

Proof of Claim 2. Since $E \subseteq B_i^p(D)$ we have $B_i^p(E) \subseteq B_i^p(B_i^p(D)) = B_i^p(D)$. So, $Z_i \subseteq B_i^p(D)$. for all $w \in Z_i$, we have $\pi(X \mid [w]_i) = r_i$.

- For each $w \in D$, we have $\pi(X \mid [w]_i) = r_i$ (by the definition of D)
- For each v, if $[v]_i \cap D \neq \emptyset$, then $\pi(X \mid [v]_i) = r_i$ (if $x \in [v]_i \cap D$, then since $x \in D$, we have $\pi(X \mid [x]_i) = r_i$ and since $x \in [v]_i$, we have $[v]_i = [x]_i$, so $\pi(X \mid [v]_i) = \pi(X \mid [x]_i) = r_i$).
- For each $v \in B_i^p(D)$, we have $[v]_i \cap D \neq \emptyset$ (otherwise, $\pi(D \mid [v]_i) = 0 \not\geq p$).
- Since $Z_i \subseteq B_i^p(D)$, we have for each $v \in Z_i$, $\pi(X \mid [v]_i) = r_i$. Hence, for each $v \in Z_i$, we have $\pi(X \cap [v]_i) = r_i \pi([v]_i)$

Then,

$$\pi(X \mid Z_i) = \pi(X \mid B_i^p(E)) = \frac{\pi(X \cap B_i^p(E))}{\pi(B_i^p(E))} = \frac{\pi\left(\bigcup_{w \in B_i^p(E)} (X \cap [w]_i)\right)}{\pi\left(\bigcup_{w \in B_i^p(E)} ([w]_i)\right)}$$

$$= \frac{\sum_{w \in B_i^p(E)} \pi(X \cap [w]_i)}{\sum_{w \in B_i^p(E)} \pi([w]_i)} = \frac{\sum_{w \in B_i^p(E)} r_i \pi([w]_i)}{\sum_{w \in B_i^p(E)} \pi([w]_i)} = r_i \frac{\sum_{w \in B_i^p(E)} \pi([w]_i)}{\sum_{w \in B_i^p(E)} \pi([w]_i)} = r_i$$

QED (of Claim)

For any Y,

$$\pi(Y \mid Z_i) = \frac{\pi(Y \cap Z_i)}{\pi(Z_i)} = \frac{\pi(Z_i \cap Z_j)}{\pi(Z_i \cap Z_j)} \cdot \frac{\pi(Y \cap Z_i)}{\pi(Z_i)} = \frac{\pi(Z_i \cap Z_j)}{\pi(Z_i)} \cdot \frac{\pi(Y \cap Z_i)}{\pi(Z_i \cap Z_j)}$$

$$\geq \frac{\pi(Z_i \cap Z_j)}{\pi(Z_i)} \cdot \frac{\pi(Y \cap Z_i \cap Z_j)}{\pi(Z_i \cap Z_j)} = \pi(Z_j \mid Z_i) \cdot \pi(Y \mid Z_i \cap Z_j)$$

Since $Z \subseteq Z_j$, we have $\pi(Z_j \mid Z_i) \geq \pi(Z \mid Z_i) \geq p$ (the latter inequality follows from Claim 1), so $\pi(Y \mid Z_i) \geq p \cdot \pi(Y \mid Z_i \cap Z_j)$.

So, for $Y = X$, we have (using Claim 2)

$$r_i = \pi(X \mid Z_i) \geq p \cdot \pi(X \mid Z_i \cap Z_j)$$

For $Y = \overline{X} = W - X$, we have

$$p(1 - \pi(X \mid Z_i \cap Z_j)) = p(\pi(\overline{X} \mid Z_i \cap Z_j)) \leq \pi(\overline{X} \mid Z_i) = 1 - \pi(X \mid Z_i) = 1 - r_i$$

Solving for r_i gives us,

$$r_i \leq 1 - p + p\pi(X \mid Z_i \cap Z_j)$$

The same argument works for r_j, so we have $r_j \geq p \cdot \pi(X \mid Z_j \cap Z_i)$ and $r_j \leq 1 - p + p \cdot \pi(X \mid Z_j \cap Z_i)$

Now, suppose $r_i \geq r_j$, then $r_i - r_j \leq (1-p) + p \cdot \pi(X \mid Z_i \cap Z_j) - p \cdot \pi(X \mid Z_j \cap Z_i) = 1 - p$. (the same holds if $r_j \geq r_i$), so we have $|r_i - r_j| \leq 1 - p$. QED