1. ***Linear time models***: A linear time model is a tuple $\mathcal{M} = \langle T, <, V\rangle$ where T is a set of **time points** (or **moments**), $< \subseteq T \times T$ is the **precedence relation**: $s < t$ ("time point occurs earlier than t") is irreflexive and transitive, and $V : \mathsf{At} \rightarrow \wp(T)$ is a valuation function (describing when the atomic propositions are true). The linear time language is given by the following grammar:

$$p \mid \neg\varphi \mid \varphi \wedge \psi \mid G\varphi \mid H\varphi$$

where $p \in \mathsf{At}$ (a countable set of atomic propositions). Truth is defined as follows:

- $\mathcal{M}, t \models p$ iff $t \in V(p)$
- $\mathcal{M}, t \models \neg\varphi$ iff $\mathcal{M}, t \not\models \varphi$
- $\mathcal{M}, t \models \varphi \wedge \psi$ iff $\mathcal{M}, t \models \varphi$ and $\mathcal{M}, t \models \psi$
- $\mathcal{M}, t \models G\varphi$ iff for all $s \in T$, if $t < s$ then $\mathcal{M}, s \models \varphi$
- $\mathcal{M}, t \models H\varphi$ iff for all $s \in T$, if $s < t$ then$\mathcal{M}, s \models \varphi$

We define $F\varphi := \neg G \neg\varphi$ and $P\varphi := \neg H \neg \varphi$, so truth for these operators is:

- $\mathcal{M}, t \models F\varphi$ iff there is $s \in T$ such that $t < s$ and $\mathcal{M}, s \models \varphi$
- $\mathcal{M}, t \models P\varphi$ iff there is $s \in T$ such that $s < t$ and $\mathcal{M}, s \models \varphi$

We say φ is valid on a temporal model $\mathcal{M} = \langle T, <, V\rangle$ provided $\mathcal{M}, t \models \varphi$ for all $t \in T$, and φ is valid on a temporal frame $\langle T, <\rangle$, provided φ is valid on every model based on $\langle T, <\rangle$ (these are standard definitions — see the notes on modal logic).

(a) A temporal frame $\langle T, <\rangle$ is **past-linear** provided for all $s, x, y \in T$, if $x < s$ and $y < s$, then either $x < y$ or $x = y$ or $y < x$. Prove that $FP\varphi \rightarrow (F\varphi \vee \varphi \vee P\varphi)$ is valid on $\langle T, <\rangle$ iff $\langle T, <\rangle$ is past-linear.

2. ***Branching-time temporal models***: Given a temporal model $\langle T, <, V\rangle$ a **branch** b is a maximal linearly ordered set of moments. We say $s \in T$ is **on a branch** b **of** T provided $s \in b$ (we also say "b is a branch going through t"). The branching time language is given by the following grammar:

$$p \mid \neg\varphi \mid \varphi \wedge \psi \mid G\varphi \mid H\varphi \mid \Box\varphi$$

where $p \in \mathsf{At}$. Truth is defined at pairs t/b where t is a moment on branch b:

- $\mathcal{M}, t/b \models p$ iff $t/b \in V(p)$
- $\mathcal{M}, t/b \models \neg\varphi$ iff $\mathcal{M}, t/b \not\models \varphi$
- $\mathcal{M}, t/b \models \varphi \wedge \psi$ iff $\mathcal{M}, t/b \models \varphi$ and $\mathcal{M}, t/b \models \psi$

- $\mathcal{M}, t/b \models G\varphi$ iff for all $s \in T$, if s is on b and $t < s$ then $\mathcal{M}, s/b \models \varphi$
- $\mathcal{M}, t/b \models H\varphi$ iff for all $s \in T$, if s is on b and $s < t$ then $\mathcal{M}, s/b \models \varphi$
- $\mathcal{M}, t/b \models \Box\varphi$ iff for all branches c through t, $\mathcal{M}, s/c \models \varphi$

For each of the following formulas, determine which are valid on all temporal frames (for those that are not valid, provide counterexamples):

(a) $\Diamond F\varphi \to F\Diamond\varphi$

(b) $\Box F\varphi \to F\Box\varphi$

(c) $F\Diamond\varphi \to \Diamond F\varphi$

(d) $F\Box\varphi \to \Box F\varphi$

3. ***Logics of Ability***: The logics of ability models of Brown are tuples $\langle W, R, V\rangle$ where $R \subseteq W \times \wp(W)$ is a relation between states and subsets of W (which Brown calls "clusters") and $V : \mathsf{At} \to \wp(W)$ a valuation function. The ability language is generated by the following grammar:

$$p \mid \neg\varphi \mid \varphi \wedge \psi \mid \llparenthesis\ \rrbracket\varphi \mid \langle\!\langle\ \rangle\!\rangle\varphi$$

where $p \in \mathsf{At}$. The intended meaning is that $\llparenthesis\ \rrbracket\varphi$ expresses "the agent is able to bring about a state where φ is true" and $\langle\!\langle\ \rangle\!\rangle\varphi$ is the weaker claim that "the agent is able to do something consistent with φ". Truth is defined as follows:

- $\mathcal{M}, w \models p$ iff $w \in V(p)$
- $\mathcal{M}, w \models \neg\varphi$ iff $\mathcal{M}, w \not\models \varphi$
- $\mathcal{M}, w \models \varphi \wedge \psi$ iff $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$
- $\mathcal{M}, t \models \llparenthesis\ \rrbracket\varphi$ iff there is a $X \subseteq W$ such that wRX and for all $v \in X$, $\mathcal{M}, v \models \varphi$
- $\mathcal{M}, t \models \langle\!\langle\ \rangle\!\rangle\varphi$ iff there is a $X \subseteq W$ such that wRX and there is a $v \in X$ such that $\mathcal{M}, v \models \varphi$

Answer the following questions:

(a) Give a counter-model to $\llparenthesis\ \rrbracket(\varphi \vee \psi) \to (\llparenthesis\ \rrbracket\varphi \vee \llparenthesis\ \rrbracket\psi)$.

(b) Prove that $\llparenthesis\ \rrbracket(\varphi \vee \psi) \to (\langle\!\langle\ \rangle\!\rangle\varphi \vee \llparenthesis\ \rrbracket\psi)$ is valid.

(c) Is $\llparenthesis\ \rrbracket\varphi \to \langle\!\langle\ \rangle\!\rangle\varphi$ valid? If it is, give a proof, and if it is not valid, give a property that would make it valid.

4. ***STIT models***: A **stit model** is a tuple $\mathcal{M} = \langle T, <, Choice, V\rangle$ where $\langle T, <, V\rangle$ is a temporal model (defined as above), and $Choice : \mathcal{A} \times T \to \wp(\wp(H_t))$ is a function mapping each agent to a partition of H_t (H_t is the set of branches going through t) satisfying the following conditions (we write $Choice_i^t$ for $Choice(i, t)$:

- $Choice_i^t \neq \emptyset$
- $K \neq \emptyset$ for each $K \in Choice_i^t$
- For all t and mappings $s_t : \mathcal{A} \to \wp(H_t)$ such that $s_t(i) \in Choice_i^t$, we have $\bigcap_{i \in \mathcal{A}} s_t(i) \neq \emptyset$

The STIT language is defined according to the following grammar:

$$\varphi \ = \ p \mid \neg\varphi \mid \varphi \wedge \psi \mid [i \text{ stit}]\varphi \mid \Box\varphi$$

where $p \in \mathsf{At}$. Truth is defined as follows:

- $\mathcal{M}, t/h \models p$ iff $t/h \in V(p)$
- $\mathcal{M}, t/h \models \neg\varphi$ iff $\mathcal{M}, t/h \not\models \varphi$
- $\mathcal{M}, t/h \models \varphi \wedge \psi$ iff $\mathcal{M}, t/h \models \varphi$ and $\mathcal{M}, t/h \models \psi$
- $\mathcal{M}, t/h \models \Box\varphi$ iff $\mathcal{M}, t/h' \models \varphi$ for all $h' \in H_t$
- $\mathcal{M}, t/h \models [i \text{ stit}]\varphi$ iff $\mathcal{M}, t/h' \models \varphi$ for all $h' \in Choice_i^t(h)$ ($Choice_i^t(h)$ is the partition cell of $Choice_i^t$ containing h)

Define $\langle i \text{ stit}\rangle\varphi$ to be $\neg[i \text{ stit}]\neg\varphi$ and $\Diamond\varphi$ to be $\neg\Box\neg\varphi$. Answer the following two questions: Suppose that there are only two agents $\mathcal{A} = \{1, 2\}$, then

(a) Prove that $\Diamond\varphi \to \langle 1 \text{ stit}\rangle\langle 2 \text{ stit}\rangle\varphi$ is valid.

(b) Conclude that $\Box\varphi$ is definable as $[1 \text{ stit}][2 \text{ stit}]\varphi$ (argue that $\Box\varphi \leftrightarrow [1 \text{ stit}][2 \text{ stit}]\varphi$ can be derived from the above axiom using the **S5** axioms for $\Box$ and $[i \text{ stit}]$, and the axiom $\Box\varphi \to [i\text{stit}]\varphi$).

The homework is DUE Tuesday, November 22 (put you answers in my mailbox).