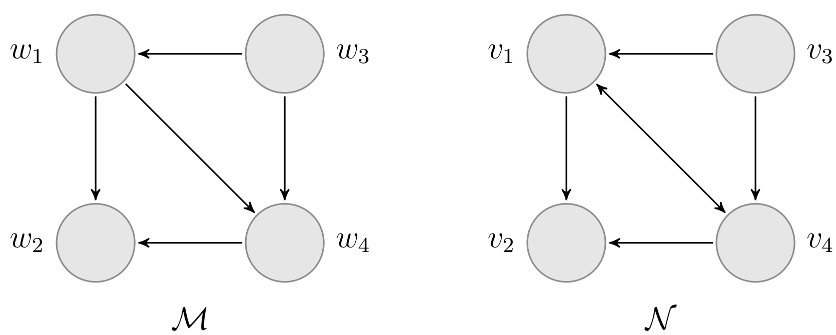
# Homework #3

(all questions have equal weight)

1. (Compare this exercise with exercise 6(a) in Section 1.2 of Enderton). Let $\mathcal{M} = \langle W, R, V\rangle$ and $\mathcal{M}' = \langle W, R, V'\rangle$ be two models that differ only in their valuation functions. Suppose that $\varphi$ is a modal formula where $V(w,p) = V'(w,p)$ for all states and sentence letters $p$ occurring in $\varphi$. Prove that $\mathcal{M} \models \varphi$ iff $\mathcal{M}' \models \varphi$. [Hint: the proof is by induction on the structure of $\varphi$.]

2. Consider the following two relational structures:




   (a) Show that the following sets are definable in $\mathcal{M}$: $\emptyset$, $\{w_1\}$, $\{w_2\}$, $\{w_3\}$, $\{w_4\}$, $\{w_1, w_2, w_3, w_4\}$.

   (b) Which of the following sets are definable in $\mathcal{N}$: $\{v_1\}$, $\{v_1, v_3\}$, $\{v_1, v_2, v_4\}$? (Note: you must explain your answers.)

3. (a) Prove that $\Diamond(P \to Q) \to (\Box P \to \Diamond Q)$ is valid.

   (b) Show that $\Box(\Box P \to Q) \vee \Box(\Box Q \to P)$ is not valid.

4. Consider the following new operator $\Box^{\leftarrow}$ defined on relational structures $\mathcal{M} = \langle W, R, V\rangle$ as follows:

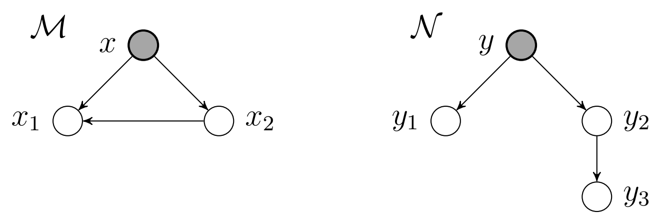
$$\mathcal{M}, w \models \Box^{\leftarrow}\varphi \text{ iff for all } v \in W, \text{ if } vRw \text{ then } \mathcal{M}, v \models \varphi$$

   Prove that $\Box^{\leftarrow}$ is not definable in the modal language.

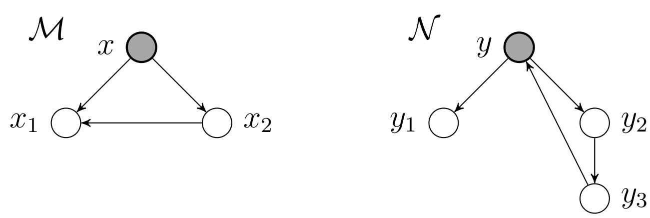
5. Is there a bisimulation connecting the states $x$ and $y$ in the following models? Justify your answer (that is, if the answer is yes, then provide a bisimulation; and if the answer is no, then find a modal formula that distinguishes the two states).

(a)

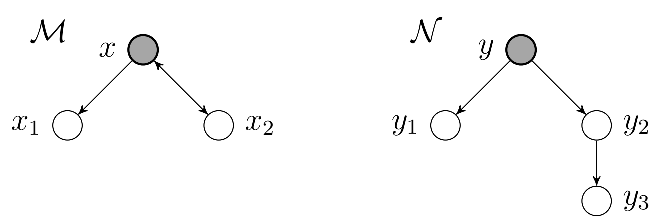

(b)




(c)




(d)




6. A frame is a pair $\langle W, R \rangle$ where $W$ is a nonempty set and $R \subseteq W \times W$ is a relation. We say that a formula is **valid on a frame** $\mathcal{F} = \langle W, R \rangle$, denote $\mathcal{F} \models \varphi$, provided $\mathcal{F}, V, w \models \varphi$ for all states $w \in W$ and valuations $V$.

   (a) We say $\mathcal{F} = \langle W, R \rangle$ is **Euclidean** if $R$ is a Euclidean relation (for all $x, y, z \in W$ if $xRy$ and $xRz$ then $yRz$). Prove that $\mathcal{F}$ is Euclidean iff $\mathcal{F} \models \Diamond\varphi \to \Box\Diamond\varphi$.

(b) We say $\mathcal{F} = \langle W, R \rangle$ is **dense** if $R$ is a dense relation (for all $x, y, z \in W$ if $xRy$, then there is a $z \in W$ such that $xRz$ and $zRy$). Prove that $\mathcal{F}$ is dense iff $\mathcal{F} \models \Box\Box\varphi \to \Box\varphi$.

**The homework is DUE Friday, January 30 at 10AM in class**.