Midterm Exam

(100 points)

Honor Code: In contrast to the homework assignments, you may **not** collaborate on this midterm exam. You may not discuss the exam with anybody but the TAs and the instructor, who will only answer clarification questions. You may use no books other than the Enderton textbook and the notes on modal logic.

1. (10 pts) Enderton, Section 1.7 #4

2. (35 pts) This question is about a sound and complete axiom system for *propositional logic*. Consider the propositional language constructed from the constant symbol $\bot$ (falsum) and implication $\to$. Consider the following formulas:

   $I1$: $P \to (Q \to P)$

   $I2$: $(P \to (Q \to R)) \to ((P \to Q) \to (P \to R))$

   $N1$: $\bot \to P$

   $N2$: $((P \to \bot) \to \bot) \to P$

   We say that φ is derivable in the axiom system $\mathbf{L}$ from a set of propositional formulas Γ, denoted $\Gamma \vdash_{\mathbf{L}} \varphi$ if there is a finite sequence of formulas $\langle \alpha_1, \ldots, \alpha_n \rangle$ with $\alpha_n = \varphi$ and for all $i \leq n$ either 1. α_i is an instance of one of the formulas $I1$, $I2$, $N1$, or $N2$, 2. $\alpha_i \in \Gamma$ or 3. α_i follows by Modus Ponens from earlier formulas. Here is an example of a formula that is derivable where Γ is empty: $\emptyset \vdash_{\mathbf{L}} A \to A$:

1.	$A \to ((X \to A) \to A)$	instance of $I1$
2.	$(A \to ((X \to A) \to A)) \to ((A \to (X \to A)) \to (A \to A))$	instance of $I2$
3.	$(A \to (X \to A)) \to (A \to A)$	MP: 1,2
4.	$A \to (X \to A)$	instance of $I1$
5.	$A \to A$	MP: 3,4

   (a) (5 pts) Prove that if $\vdash_{\mathbf{L}} \varphi \to \psi$ and $\vdash_{\mathbf{L}} \psi \to \chi$ then $\vdash_{\mathbf{L}} \varphi \to \chi$ (here you cannot appeal to soundness, completeness or the deduction theorem discussed in class — you must provide a deduction of $\varphi \to \chi$ given that $\varphi \to \psi$ and $\psi \to \chi$ are deducible).

   Recall that $\Gamma \models \varphi$ means that if v is any truth assignment satisfying all formulas in Γ, then v satisfies φ.

   (b) (10 pts) Prove soundness of this axiom system: if $\Gamma \vdash_{\mathbf{L}} \varphi$ then $\Gamma \models \varphi$.

For completeness, we can follow the strategy discussed in class. Call a set Γ of propositional formula **L-consistent** if $\Gamma \not\vdash_{\mathbf{L}} \bot$ and **L**-inconsistent if Γ is not consistent. A set Γ is said to be **maximally L-consistent** if Γ is **L**-consistent and for all Γ' with $\Gamma \subsetneq \Gamma'$ (i.e., Γ' is a *strict* superset of Γ), Γ' is **L**-inconsistent.

In what follows, you can use the **deduction theorem**: if Γ is a set of formulas and α and β are formulas, then $\Gamma; \alpha \vdash_{\mathbf{L}} \beta$ iff $\Gamma \vdash_{\mathbf{L}} \alpha \to \beta$. The proof of this fact was sketched in class. (*Note that you* **cannot** *used this theorem for part (a)*).

(c) (10 pts) Prove that for all maximally **L**-consistent sets Γ we have (**Hint**: before proceeding, prove (assuming the deduction theorem) that if Γ is a maximally **L**-consistent set, then $\Gamma \vdash_{\mathbf{L}} \varphi$ iff $\varphi \in \Gamma$.)

   i. $\varphi \in \Gamma$ iff $\varphi \to \bot \notin \Gamma$

   ii. $\varphi \to \psi \in \Gamma$ iff $\varphi \notin \Gamma$ or $\psi \in \Gamma$ (**Hint**: use (a) for the right to left direction)

   (*Note that you must prove these two fact directly and cannot simply appeal to the corresponding facts about maximally consistent sets used for proving completeness of the minimal modal logic.*)

The proof of Lindenbaum's Lemma is exactly as it was presented in class. Furthermore, one can prove a Truth Lemma:

(d) (5 pts) Give any maximally **L**-consistent set Γ, prove there is a truth assignment v such that for all propositional formulas φ (built using $\to$ and $\bot$), $\varphi \in \Gamma$ iff $\overline{v}(\varphi) = T$. (Hint: the truth assignment used in Section 1.7 for the proof the compactness theorem works here as well.)

Conclude the proof by showing that

(e) (5 pts) If $\Gamma \models \varphi$ then $\Gamma \vdash_{\mathbf{L}} \varphi$.

3. (35 pts) We say a frame $\langle W, R\rangle$ is **secondary reflexive** of $\forall x \forall y (xRy \to yRy)$ is true in $\langle W, R\rangle$ (here we view the frame $\langle W, R\rangle$ as a first order structure with domain W and R a relation on W).

   (a) (10 pts) Prove that for all frames $\mathcal{F} = \langle W, R\rangle$, $\mathcal{F} \models \Box(\Box\varphi \to \varphi)$ iff $\mathcal{F}$ is secondary reflexive.

   Let $\mathbf{R}$ be the axiom system that extends the minimal modal logic $\mathbf{K}$ with all instances of $\Box(\Box p \to p)$. That is we write $\vdash_{\mathbf{R}} \varphi$ if there is a finite sequence of formulas $\langle \alpha_1, \ldots, \alpha_n\rangle$ with $\alpha_n = \varphi$ and for all $i \leq n$ either 1. α_i is a tauotology, 2. α_i is an instance of $\Box(p \to q) \to (\Box p \to \Box q)$, 3. α_i is an instance of $\Box(\Box p \to p)$, or 4. α_i follows from earlier formulas by either Modus Ponens or Necessitation. Suppose that $\mathbb{F}$ is the class of all secondary reflexive frames:

$$\mathbb{F} = \{\mathcal{F} = \langle W, R\rangle \mid \mathcal{F} \text{ is secondary reflexive } \}$$

We say φ is valid on $\mathbb{F}$, denoted $\models_{\mathbb{F}} \varphi$ provided $\mathcal{F} \models \varphi$ for all $\mathcal{F} \in \mathbb{F}$. In this exercise we will prove that **R** is sound and complete with respect to $\mathbb{F}$:

$$\vdash_{\mathbf{R}} \varphi \text{ iff } \models_{\mathbb{F}} \varphi$$

First a warm-up exercise:

(b) (10 pts) For each of the following formulas determine whether or not they are valid on $\mathbb{F}$. If the formula is valid, then provide a deduction in the logic **R**, otherwise find a model where the formula is false (recall that $\Diamond\varphi$ is defined to be $\neg\Box\neg\varphi$).

i. $\Box r \to \Box\Diamond r$

ii. $\Diamond r \to \Box\Diamond r$

Soundness is an easy extension of soundness of **K** for the class of all frames: one need only check that all instances of $\Box(\Box p \to p)$ are valid on $\mathbb{F}$, which follows from part (a). For completeness, we proceed as we did in class. The **canonical model** for the axiom system **R** is $\mathcal{M}_{\mathbf{R}} = \langle W_{\mathbf{R}}, R_{\mathbf{R}}, V_{\mathbf{R}} \rangle$ where $W_{\mathbf{R}} = \{\Gamma \mid \Gamma \text{ is a maximal } \mathbf{R}\text{-consistent set }\}$ and $R_{\mathbf{R}}$ and $V_{\mathbf{R}}$ are defined as in class. The Lindenbaum Lemma (every **R**-consistent set can be extended to a maximally **R**-consistent set) and Truth Lemma (for all modal formula φ, $\mathcal{M}_{\mathbf{R}}, \Gamma \models \varphi$ iff $\varphi \in \Gamma$) are proven exactly as in class. (see also Chapter 5 of *Modal Logic for Open Minds*). The only thing that needs to be proven is that $\mathcal{M}_{\mathbf{R}}$ is the "right kind of model":

(c) (10 pts) Prove that $\langle W_{\mathbf{R}}, R_{\mathbf{R}} \rangle$ is a secondary reflexive frame.

(d) (5 pts) Conclude that $\models_{\mathbb{F}} \varphi$ implies $\vdash_{\mathbf{R}} \varphi$.

4. (10 pts) Enderton, Section 2.2 #6

5. (10 pts) Enderton, Section 2.2 #9

The midterm is DUE Friday, February 13 at 10AM in class.