Background: Logic and Probability

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1 Basic Propositional Logic

Syntax: A formula of propositional logic is constructed as follows:

- If p is an **atomic propositions**, then p is a formula.
- If P is a formula, then so is $\neg P$ (read "not P")
- If P and Q are formulas, then so is $P \wedge Q$ (read "P and Q")
- If P and Q are formulas, then so is $P \vee Q$ (read "P or Q")
- If P and Q are formulas, then so is $P \rightarrow Q$ (read "P implies Q" or "if P, then Q")

Let $Form$ be the set of all formulas.

Semantics: The **truth value** of an formula can be calculated from the following truth tables:

P	$\neg P$
T	F
F	T

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Example Question. There are three suspects for a murder: Adams, Brown and Clark. Adams says "I didn't do it. The victim was an old acquaintance of Brown's. But Clark hated him." Brown states "I didn't do it. I didn't even know the guy. Besides I was out of town all that week." Clark says "I didn't do it. I saw both Adams and Brown downtown with the victim that day; one of them must have done it." Assume that the two innocent men are telling the truth, but that the guilty man might not be. Who did it?

Answer. Consider three sentence symbols A, B and C with the following intended meanings:

A: "Adams speaks the truth"

B: "Brown speaks the truth"

C: "Clark speaks the truth"

We have the following information from the story:

1. There are two innocent men and they are telling the truth: $(A \wedge B) \vee (B \wedge C) \vee (A \wedge C)$

2. If Adams tells the truth, then Brown cannot be: $A \to \neg B$

3. If Brown tells the truth, both Adams and Clark must be lying: $B \to (\neg A \wedge \neg C)$

4. If Clarke tells the truth, Brow must be lying: $C \to \neg B$.

All 4 statements must be true simultaneously which leaves only one possibility (the gray row):

A	B	C	$(A \wedge B) \vee (B \wedge C) \vee (A \wedge C)$	$A \to \neg B$	$B \to (\neg A \wedge \neg C)$	$C \to \neg B$
F	F	F	F	T	T	T
F	F	T	F	T	T	T
F	T	F	F	T	T	T
F	T	T	T	T	F	F
T	F	F	F	T	T	T
T	F	T	T	T	T	T
T	T	F	T	F	F	T
T	T	T	T	F	F	F

Hence, Adams and Clark are truthful and Brown is not, so Brown is the killer.

Rules of Inference: A **rule of inference** is a function taking a sequence of formulas (the premises) and returning a formula (the conclusion), denoted $P_1, \ldots, P_n \vdash Q$. An inference rule is **(classically) valid** if "any way of making the premises true also makes the conclusion true". The following are classically valid rules of inference:

- *Modus Ponens*: P, $P \to Q \vdash Q$
- *Modus Tollens*: $\neg Q$, $P \to Q \vdash \neg P$
- *Disjunctive Syllogism*: $P \vee Q$, $\neg P \vdash Q$
- *Adjunction*: P_1, P_2,$\ldots P_n \vdash P_1 \wedge \cdots \wedge P_n$
- *Noncontradiction*: P, $\neg P \vdash Q$
- *Monotonicity*: $P \to Q \vdash (P \wedge R) \to Q$;
  $P \vdash Q$ implies $P, R \vdash Q$

Make sure you can explain why each of the above rules are valid.

2 Probability

Probability Measure. Suppose that W is a finite set of states. A probability measure assigns real values between 0 and 1 to subsets of W. That is, a **probability measure** is a function $p : \mathcal{E} \to [0, 1]$ where $\mathcal{E}$ is the set of events (all subsets of W) that satisfies the Komogorov axioms.

Kolmogorov Axioms:

1. For each $E \in \mathcal{E}$, $0 \leq p(E) \leq 1$
2. $p(W) = 1$, $p(\emptyset) = 0$
3. If $E_1, \ldots, E_n, \ldots$ are pairwise disjoint ($E_i \cap E_j = \emptyset$ for $i \neq j$), then $p(\bigcup_i E_i) = \sum_i p(E_i)$

Any probability measure p satisfies the following properties (*make sure you can explain why*)

- $p(\overline{E}) = 1 - p(E)$ ($\overline{E}$ is the complement of E)
- If $E \subseteq F$ then $p(E) \leq p(F)$
- $p(E \cup F) = p(E) + p(F) + p(E \cap F)$

Conditional Probability: The probability of E *given* F, dented $p(E|F)$, is defined to be

$$p(E|F) = \frac{p(E \cap F)}{p(F)}.$$

Bayes Theorem: $p(E|F) = p(F|E)\frac{p(E)}{p(F)}$

Bayes theorem is important because it expresses the quantity $p(E|F)$ (the probability of a hypothesis E given the evidence F) — which is something people often find hard to assess — in terms of quantities that can be drawn directly from experiential knowledge. For example, suppose you are in a casino and you hear a person at the next gambling table announce "Twelve". We want to know wether he was rolling a pair of dice or a roulette wheel. That is, compare $p(\mathit{Dice} \mid \mathit{Twelve})$ with $p(\mathit{Roulette} \mid \mathit{Twelve})$. Based on our background knowledge of gambling we have $p(\mathit{Twelve} \mid \mathit{Dice}) = 1/36$ and $p(\mathit{Twelve} \mid \mathit{Roulette}) = 1/38$. Based on our observations about the casino, we can judge the prior probabilities $p(\mathit{Dice})$ and $p(\mathit{Roulette})$. But this is now enough to *calculate* the required probabilities.

Question: The Three Prisoners Paradox. Three prisoners A, B and C have been tried for murder and their verdicts will told to them tomorrow morning. They know only that one of them will be declared guilty and will be executed while the others will be set free. The identity of the condemned prisoner is revealed to the very reliable prison guard, but not to the prisoners themselves. Prisoner A asks the guard "Please give this letter to one of my friends — to the one who is to be released. We both know that at least one of them will be released".

An hour later, A asks the guard "Can you tell me which of my friends you gave the letter to? It should give me know clue regarding my own status because, regardless of my fate, each of my friends had an equal chance of receiving my letter." The guard told him that B received his letter.

Prisoner A then concluded that the probability that he will be released is 1/2 (since the only people without a verdict are A and C). But, A thinks to himself:

"Before I talked to the guard my chance of being executed was 1 in 3. Now that he told me B has been released, only C and I remain, so my chances of being executed have gone from 33.33% to 50%. What happened? I made certain not to ask for any information relevant to my own fate..."

Explain what is wrong with A's reasoning.

Answer. A reasoned as follows. Consider the following events:

G_A: "Prisoner A will be declared guilty" (we have $p(G_A) = 1/3$)

I_B: "Prisoner B will be declared innocent" (we have $p(I_B) = 2/3$)

We have $p(I_B \mid G_A) = 1$: "If A is declared guilty then B will be declared innocent." Using Bayes Theorem,

$$p(G_A \mid I_B) = p(I_B \mid G_A)\frac{p(G_A)}{p(I_B)} = 1 \cdot \frac{1/3}{2/3} = 1/2$$

But, A did not receive the information that B will be declared innocent, but rather that "the guard said that B will be declared innocent." So, A should have conditioned on the event:

I'_B: "The guard said that B will be declared innocent"

Given that $p(I'_B \mid G_A)$ is 1/2 (given that A is guilty, there is a 50-50 chance that the guard could have given the letter to B or C). This gives us the following correct calculation:

$$p(G_A \mid I'_B) = p(I'_B \mid G_A)\frac{p(G_A)}{p(I'_B)} = 1/2 \cdot \frac{1/3}{1/2} = 1/3$$

3 Something to Think About

1. **The Birthday Paradox**: What is the probability of two or more people out of a group of n do have the same birthday? This is not a paradox but a result that people often find puzzling. Of course, if there are 367 people in the room, then there *must* be two people that share the same birthday. What is surprising is that if there are only 23 people in the room, then there is about a 50% chance that two people have the same birthday and with only 75 people the probability goes up to 99.9% that two people have the same birthday (why?).

2. **Problems from *Rational Choice* by I. Gilboa**

   (a) Explain what is wrong with the claim, Most good chess players are Russian; therefore a Russian is likely to be a good chess player.

   (b) Comment on the claim, Some of the greatest achievements in economics are due to people who studied mathematics. Therefore, all economists had better study mathematics first.

   (c) 5. Trying to understand why people confuse $P(A \mid B)$ and $P(B \mid A)$, it is useful to see that qualitatively, if A makes B more likely, it will also be true that B will make A more likely.

i. Show that for any two events A, B

$P(A \mid B) > P(A \mid \overline{B})$
iff
$P(A \mid B) > P(A) > P(A \mid \overline{B})$
iff
$P(B \mid A) > P(B \mid \overline{A})$
iff
$P(B \mid A) > P(B) > P(B \mid \overline{A})$

where $\overline{A}$ is the complement of A. (Assume that all probabilities involved are positive, so that all the conditional probabilities are well defined.)

ii. If the proportion of Russians among the good chess players is higher than their proportion overall in the population, what can be said?

3. **The Surprise Examination Paradox**: This paradox that has been widely discussed by logicians and philosophers (there still is no general agreement about its "solution").

> A teacher announces in class that an examination will be held on some day during the following week, and moreover that the examination will be a surprise. The students argue that a surprise exam cannot occur. For suppose the exam were on the last day of the week. Then on the previous night, the students would be able to predict that the exam would occur on the following day, and the exam would not be a surprise. So it is impossible for a surprise exam to occur on the last day. But then a surprise exam cannot occur on the penultimate day, either, for in that case the students, knowing that the last day is an impossible day for a surprise exam, would be able to predict on the night before the exam that the exam would occur on the following day. Similarly, the students argue that a surprise exam cannot occur on any other day of the week either. Confident in this conclusion, they are of course totally surprised when the exam occurs (on Wednesday, say). The announcement is vindicated after all. Where did the students' reasoning go wrong?

4. **Monty Hall Puzzle**: The original formulation (from *Ask Marilyn* in *Parade Magazine*) is

> Suppose you're on a game show, and you're given the choice of three doors. Behind one door is a car, behind the others, goats. You pick a door, say number 1, and the host, who knows what's behind the doors, opens another door, say number 3, which has a goat. He says to you, "Do you want to pick door number 2?" Is it to your advantage to switch your choice of doors?

5. **The Two-Envelop Paradox**: This paradox is similar to the Monty Hall Puzzle, though much more difficult. See this link

`http://www.maa.org/devlin/devlin_0708_04.html`

for a nice discussion of the paradox and a proposed solution.

You are presented with two indistinguishable envelopes containing some money. You are further informed that one of the envelopes contains twice as much money as the other. You may select any one of the envelopes and you will receive the money in the selected envelope. When you have selected one of the envelopes at random but not yet opened it, you get the opportunity to take the other envelope instead. Should you switch to the other envelope? (There is a convincing argument that you should switch, but then ask yourself, *should you switch again*?)